Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 203: Calculus with Analytic Geometry III

Final Exam: Solutions and Answers

Instructor:

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Problem 1. Consider the points $A = (1, 4, 2)$, $B = (-1, 3, 5)$ and $C = (3, 5, 7)$ in the three-dimensional linear space $\mathbb{R}^3$.

(1). The fourth point D is defined by the condition that the points A, B, C, D are the consecutive vertices of a certain parallelogram. Find the coordinates of the point D.

Space for your solution:

If A, B, C, D are consecutive vertices of a parallelogram, then

$$D = B + \overrightarrow{BD} = B + \overrightarrow{BA} + \overrightarrow{BC} = B + (A - B) + (C - B) =$$

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \right) + \left(\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \right) =$$

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}.$$

(2). Give an **explicit** equation that defines the plane in which the parallelogram $ABCD$ lies.

Space for your solution:

Slightly generalizing the previous consideration and denoting P an arbitrary point of the plane in question, we get

$$P = B + \overrightarrow{BD} = B + \lambda \cdot \overrightarrow{BA} + \mu \cdot \overrightarrow{BC}$$

with $\lambda, \mu \in \mathbb{R}$.

In coordinates, this yelds:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + \lambda \cdot \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \mu \cdot \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

(3). Compute the area of the parallelogram $ABCD$.

Space for your solution:

One possible way to do it is to compute

$$\text{Area} = |\overrightarrow{BA}| \cdot |\overrightarrow{BC}| \cdot \sin \widehat{ABC}$$

using dot product to find $|\overrightarrow{BA}|$,$|\overrightarrow{BC}|$, and $\cos \widehat{ABC}$. Then we would need to use the Pythagorean identity $(\cos\theta)^2 + (\sin\theta)^2 = 1$ to get the (positive value) of $\sin \widehat{ABC}$.

Another way is to compute the length of the cross product:

$$\text{Area} = \left|\overrightarrow{BA} \times \overrightarrow{BC}\right| =$$

$$\left|\det \begin{bmatrix} \vec{i} & 2 & 4 \\ \vec{j} & 1 & 2 \\ \vec{k} & -3 & 2 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \cdot \vec{i} - \det \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix} \cdot \vec{j} + \det \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot \vec{k}\right| =$$

$$\left|\big(1 \cdot 2 - 2 \cdot (-3)\big) \cdot \vec{i} - \big(2 \cdot 2 - 4 \cdot (-3)\big) \cdot \vec{j} + \big(2 \cdot 2 - 4 \cdot 1\big) \cdot \vec{k}\right| =$$

$$\left|8\,\vec{i} - 16\,\vec{j} + 0\,\vec{k}\right| = \sqrt{8^2 + 16^2} = \sqrt{64 + 256} = \sqrt{320} = 8\sqrt{5}.$$

(4). Find the distance from the point $P = (1, 1, 1)$ to the plane that passes through the points A, B, C. (Hint: compute the volume of the parallelepiped generated by the vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$, and $\overrightarrow{BP}$. Then use the area found in the previous subproblem.)

Space for your solution:

The volume can be computed using determinants:

$$\text{Volume} = \left|\det \begin{bmatrix} 1-(-1) & 2 & 4 \\ 1-3 & 1 & 2 \\ 1-5 & -3 & 2 \end{bmatrix}\right| =$$

$$\left|\det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \cdot 2 - \det \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix} \cdot (-2) + \det \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot (-4)\right| =$$

$$\left|\big(1 \cdot 2 - 2 \cdot (-3)\big) \cdot 2 - \big(2 \cdot 2 - 4 \cdot (-3)\big) \cdot (-2) + \big(2 \cdot 2 - 4 \cdot 1\big) \cdot (-4)\right| =$$

$$|8 \cdot 2 - 16 \cdot (-2) + 0 \cdot (-4)| = 16 + 32 = 48.$$

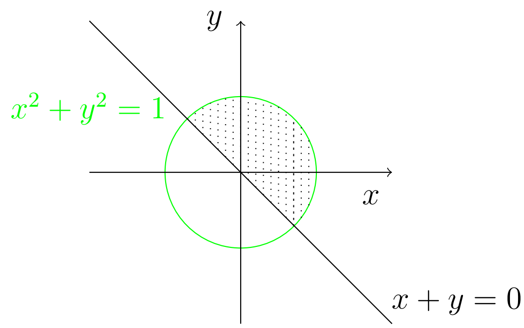
The distance h we are looking for is the height of the parallelepiped under consideration. Thus, Volume $= h \cdot$ Area, where Area is the base area computed in the previous problem. Therefore $h = \frac{\text{Volume}}{\text{Area}} = \frac{48}{8\sqrt{5}} = \frac{6\sqrt{5}}{5}$.

Problem 2. The region R on the (x, y)-plane is given as the solution set of the system

$$\begin{cases} x^2 + y^2 \le 1 \\ x + y \ge 0 \end{cases}$$

(1). Sketch this region in (x, y)-coordinate system.

Space for your solution:



(2). Find a system of equations or inequalities that describes all the **interior** points of the region R relative to the (x, y)-plane.

Space for your solution:

$$\begin{cases} x^2 + y^2 < 1 \\ x + y > 0 \end{cases}$$

(3). Find a system of equations or inequalities that describes all the **boundary** points of the region R relative to the (x, y)-plane.

Space for your solution:

$$\left[\begin{array}{l} \begin{cases} x^2 + y^2 = 1 \\ x + y \ge 0 \end{cases} \\ \\ \begin{cases} x^2 + y^2 \le 1 \\ x + y = 0 \end{cases} \end{array} \right.$$

(4). Find all the critical points of the function $f(x, y) = y + x^2$ which are interior to the region R.

Space for your solution:

The differential

$$\mathtt{d}f = \frac{\partial f}{\partial x}\,\mathtt{d}x + \frac{\partial f}{\partial y}\,\mathtt{d}y = \frac{\partial(y + x^2)}{\partial x}\,\mathtt{d}x + \frac{\partial(y + x^2)}{\partial y}\,\mathtt{d}y = 2x\,\mathtt{d}x + \,\mathtt{d}y$$

is never zero since its y component is constant 1. Therefore, there are no critical points of f in the interior of the region R.

(5). Find all the critical points of the function $f(x, y) = y + x^2$ on the boundary of the region R.

Space for your solution:

The boundary of R has two components: the interval and the arc. Parameterizing the interval with $\gamma(x) = (x, -x)$ for $x \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, we get a function

$$f\big(\gamma(x)\big) = -x + x^2$$

whose derivative $2x - 1$ equals zero for $x = \frac{1}{2}$. The corresponding critical point on the boundary is

$$\gamma\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Parameterizing the arc with $\omega(t) = \big(\cos(t), \sin(t)\big)$ for $t \in [-\frac{\pi}{4}, \frac{3\pi}{4}]$, we get a function

$$f\big(\omega(t)\big) = \sin(t) + \cos^2(t)$$

whose derivative $\cos(t) - 2\cos(t)\sin(t)$ equals zero for $t = \frac{\pi}{2}$ and $t = \frac{\pi}{6}$. The corresponding critical points on the boundary are

$$\omega\left(\tfrac{\pi}{2}\right) = (0, 1) \text{ and } \omega\left(\tfrac{\pi}{6}\right) = \left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right).$$

In addition, we need to add the corners of the region R:

$$\left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right) \text{ and } \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)$$

to the list of critical points on the boundary of region R.

(6). Find the maximum and the minimum values of the function $f(x, y) = y + x^2$ on the region R, and the corresponding points of maxima and minima.

Space for your solution:

Evaluating the function $f(x, y) = y + x^2$ in all critical points, we get:

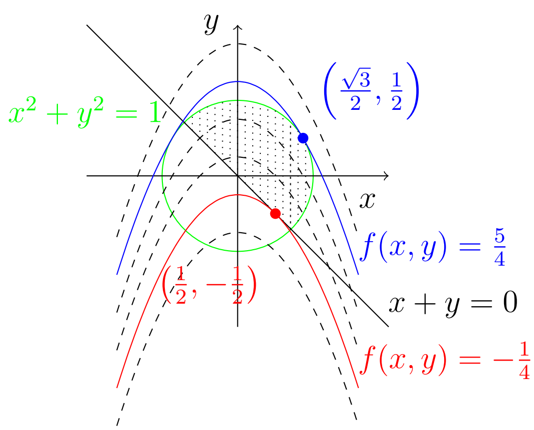
(x, y)	$f(x, y)$
$\left(\frac{1}{2}, -\frac{1}{2}\right)$	$-\frac{1}{4}$
$(0, 1)$	1
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{5}{4}$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$\frac{1-\sqrt{2}}{2}$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{1+\sqrt{2}}{2}$

The maximum among these values of the function is $\frac{5}{4}$, making $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ the (only) point of global maximum of f on region R.

The minimum among these values of the function is $-\frac{1}{4}$, making $\left(\frac{1}{2}, -\frac{1}{2}\right)$ the (only) point of global minimum of f on region R.

(7). Draw the contour diagram of the function $f(x, y) = y + x^2$ to confirm your work above.

Space for your solution:



Problem 3. Compute the work of the force $\overrightarrow{F} = -\overrightarrow{k}$ when moving along the path γ parameterized by

$$\begin{cases} x = t \\ y = \cos(t) \\ z = \sin(t) \end{cases}$$

with $t \in [0, \frac{\pi}{2}]$.

Space for your solution:

$$\text{Work} = \int_\gamma \overrightarrow{F} \cdot \, \mathtt{d}\overrightarrow{r} =$$

$$= \boxed{\text{note that this vector field is potential: } \overrightarrow{F} = \text{grad } (-z)} =$$

$$= \Big(-\sin\left(\frac{\pi}{2}\right)\Big) - \Big(-\sin(0)\Big) = -1.$$