Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 203, CRN 95613
Calculus with Analytic Geometry III

Final Exam: Solutions and Answers

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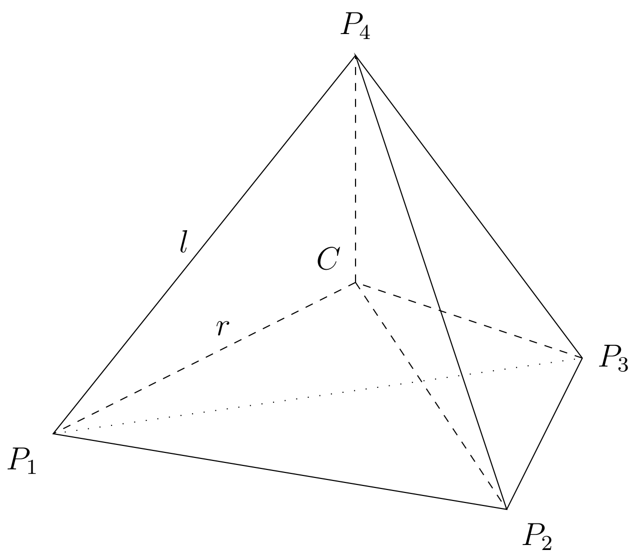
Problem 1. There are several lessons that may be drawn from solving this problem. Placing the situation at hand into a larger context may give additional tools for analyzing the situation. Considering the same situation in simpler circumstances (say, lower dimension) may give additional insight. A good choice of a system of coordinates is an essential solution step. A good system of coordinates should reflect the symmetries of the problem itself.

Suppose four points P_1, ... ,P_4 are positioned in a three dimensional space $\mathbb{R}^3$ in such a way that all the distances between them are equal: $\exists l \in \mathbb{R} : \forall i = 1 \dots 4, \forall j = 1 \dots 4 :$

$$i \neq j \;\Rightarrow\; |P_i P_j| = l.$$

In other words, these four points are the vertices of a regular tetrahedron and l is the length of its edge.

There exists unique fifth point $C \in \mathbb{R}^3$ equidistant (denote that distance r) from these four points. In other words, C is the center of the tetrahedron:



(1). Introduce a system of coordinates in a four dimensional space $\mathbb{R}^4$ in such a way that the coordinates of all the points P_1, ... ,P_4 are symmetric with respect to that system of coordinates. Hint: try to generalize from lower-dimensional cases. When you have two points P_1 and P_2 in a one dimensional space $\mathbb{R}$, you can place this $\mathbb{R}$ into $\mathbb{R}^2$ as the line $x+y=1$, so that $P_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $P_2 = \begin{bmatrix}0\\1\end{bmatrix}$, and $C = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$. When you have three points P_1, P_2 and P_3 in a two dimensional space $\mathbb{R}^2$, you can place this $\mathbb{R}^2$ into $\mathbb{R}^3$ as the plane $x+y+z=1$, so that $P_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $P_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $P_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, and $C = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$.
What are the coordinates of P_1, ... ,P_4 and C in this system of coordinates?

Space for your solution:

$$P_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, P_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, P_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, P_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \text{ and } C = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

(2). Express the distance r from the point C to any of the points P_i in terms of the distance l between any of the two distinct points P_i, P_j. Hint: find two distances – between P_1 and P_2 and P_1 and C using coordinates introduced in the previous problem.

Space for your solution:

$$l = dist(P_1, P_2) = |\overrightarrow{P_1P_2}| = \sqrt{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_2}} = \sqrt{\overrightarrow{P_2 - P_1} \cdot \overrightarrow{P_2 - P_1}} =$$

$$\sqrt{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} = \sqrt{(-1)^2 + (1)^2 + (0)^2 + (0)^2} = \sqrt{1+1+0+0} = \sqrt{2}$$

$$r = dist(C, P_1) = |\overrightarrow{CP_1}| = \sqrt{\overrightarrow{CP_1} \cdot \overrightarrow{CP_1}} = \sqrt{\overrightarrow{P_1 - C} \cdot \overrightarrow{P_1 - C}} =$$

$$\sqrt{\begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}} = \sqrt{\left(\frac{3}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2}$$

$$= \sqrt{\frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2}$$

$$r = \frac{r}{l} \cdot l = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} \cdot l = \sqrt{\frac{3}{8}} \cdot l$$

(3). Find the angle between $\overrightarrow{CP_i}$ and $\overrightarrow{CP_j}$ for $i, j = 1 \ldots 4, i \neq j$.

Space for your solution:

Using the dot product again, we can compute

$$\widehat{P_1CP_2} = \arccos\left(\frac{\overrightarrow{CP_1} \cdot \overrightarrow{CP_2}}{|\overrightarrow{CP_1}| \cdot |\overrightarrow{CP_2}|}\right) = \arccos\left(\frac{(P_1 - C) \cdot (P_2 - C)}{|P_1 - C| \cdot |P_2 - C|}\right) =$$

$$\arccos\left(\frac{\begin{bmatrix}\frac{3}{4}\\-\frac{1}{4}\\-\frac{1}{4}\\-\frac{1}{4}\end{bmatrix} \cdot \begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}\\-\frac{1}{4}\\-\frac{1}{4}\end{bmatrix}}{\left|\begin{bmatrix}\frac{3}{4}\\-\frac{1}{4}\\-\frac{1}{4}\\-\frac{1}{4}\end{bmatrix}\right| \cdot \left|\begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}\\-\frac{1}{4}\\-\frac{1}{4}\end{bmatrix}\right|}\right) =$$

$$\arccos\left(\frac{\left(\frac{3}{4}\right) \cdot \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right) \cdot \left(\frac{3}{4}\right) + \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{4}\right)}{\left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)}\right) =$$

$$\arccos\left(\frac{-\frac{3}{16} - \frac{3}{16} + \frac{1}{16} + \frac{1}{16}}{\left(\frac{3}{4}\right)}\right) = \arccos\left(\frac{\left(-\frac{1}{4}\right)}{\left(\frac{3}{4}\right)}\right) = \arccos\left(-\frac{1}{3}\right) \approx 1.91 \approx 109.47^\circ.$$

Problem 2.

Recall that the *orientation* of a space is defined by a choice of basis in that space. Two bases define the same orientation if and only if one of them can be continuously deformed into another while staying a basis. (More precisely, orientation is an equivalence class of bases, where the equivalence is defined as the possibility of deforming one basis into another continuously within the set of bases.) The special case of a zero-dimensional space necessitates separate definition. Orientation of a single point is the choice of coefficient $+1$ or -1 for that point. (There is also a more abstract way to define orientation that does not need to treat zero dimension as a special case.)

(1). Does the basis $\vec{i}, \vec{j}$ define the same orientation on $\mathbb{R}^2$ as $\vec{j}, \vec{i}$?

Space for your solution:

No. Rotation of basis $\vec{i}, \vec{j}$ by $\frac{\pi}{2}$ continuously deforms it into the basis $\vec{j}, -\vec{i}$ (while keeping it linearly independent). Thus $\vec{i}, \vec{j}$ can be continuously deformed into $\vec{j}, \vec{i}$ if and only if $\vec{j}, -\vec{i}$ can be continuously deformed into $\vec{j}, \vec{i}$. But the line generated by $\vec{j}$ splits the plane into two disjoint components. A vector cannot continuously change from $\vec{i}$ to $-\vec{i}$ without crossing that line and becoming linearly dependent with vector $\vec{j}$.

(2). Does the basis $\vec{i}, \vec{j}, \vec{k}$ define the same orientation on $\mathbb{R}^3$ as $\vec{j}, \vec{k}, \vec{i}$?

Space for your solution:

Yes, because we can rotate the standard basis as a rigid whole by $\frac{\pi}{3}$ around $\vec{i} + \vec{j} + \vec{k}$, so that each vector turns into the next one, and the last becomes the first.

(3). How many orientations are there on an arbitrary space?

Space for your solution:

A linear space may have only two orientations. Zero dimensinal case is special and has two orientations defined by the choice of $+$ or $-$ sign.

Suppose the dimension n of the linear space is at least 1, and $e_1, \ldots, e_{n-1}, e_n$ is an arbitrary basis. Then one orientation is given by $e_1, \ldots, e_{n-1}, e_n$, and the other one by $e_1, \ldots, e_{n-1}, -e_n$.

Indeed, for any other basis $e'_1, \ldots, e'_{n-1}, e'_n$, vectors $e_1, \ldots, e_{n-1}$ can be continuously deformed into $e'_1, \ldots, e'_{n-1}$ with the whole system of n vectors remaining linearly independent (why?). At the end of this deformation, the e_n ends up becomes some other vector e_*. The subspace generated by $e'_1, \ldots, e'_{n-1}$ separates $\mathbb{R}^n$ into two connected components. If e'_n and e_* are in the same component, then the orientations given by $e_1, \ldots, e_{n-1}, e_n$ and by $e'_1, \ldots, e'_{n-1}, e'_n$ are the same. If they are in different components, then e_* has no possibility of deforming into e'_n without crossing the subspace generated by $e'_1, \ldots, e'_{n-1}$, thus making $e'_1, \ldots, e'_{n-1}, e_*$ linearly dependent.

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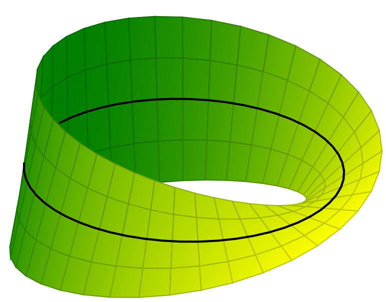
The above is the only part of the answer I hope to get from you on the exam.

A general space (like the pair of endpoints of a path, a curve or a surface) presents two difficulties for defining orientation. The first one is that the space itself can be curved, so the notion of a basis would not directly apply to it. This difficulty can be resolved by choosing a basis *tangent* to the original space. The second difficulty is the possibility of the space having several connected components. We address it by selecting one basis in every component. When deforming bases into each other to check whether they define the same orientation, we may have to move them along the space, and not just inside of their respective tangent spaces.

A connected space may have one (then it is called *non-orientable*) or two (then it is called *orientable*) orientations. An example of a non-orientable space — i.e. a space with only one orientation — is the Möbius band:



The number of orientations a general space may have is 2^n, where n is the number of its orientable connected components. This is because orientation can be chosen independently among the two choices for each one of those components.

Problem 3.

Consider two vectors v and w on a plane $\mathbb{R}^2$ with the standard orientation. Define the *wedge product* of these two vectors, denoted $v \wedge w$, as

$$v \wedge w = \begin{cases} 0 : \text{if the vectors } v \text{ and } w \text{ are linearly dependent;} \\ (\text{area of the parrallelogram, generated by } v \text{ and } w) : \\ \quad \text{if } v, w \text{ is a basis of } \mathbb{R}^2 \text{ defining its standard orientation;} \\ -(\text{area of the parrallelogram, generated by } v \text{ and } w) : \\ \quad \text{if } v, w \text{ is a basis of } \mathbb{R}^2 \text{ defining the orientation, opposite to its standard one.} \end{cases}$$

(1). Determine the result of $v \wedge w$ when v and w are the standard basis vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ in any combination.

Space for your solution:

$\overrightarrow{i} \wedge \overrightarrow{i} = 0$, $\overrightarrow{i} \wedge \overrightarrow{j} = 1$, $\overrightarrow{j} \wedge \overrightarrow{j} = 0$, $\overrightarrow{j} \wedge \overrightarrow{i} = -1$.

(2). If $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\overrightarrow{i}, \overrightarrow{j}$, compute the value of $v \wedge w$ in terms of the $a_{i,j}$. Use (without proof) the fact that $v \wedge w$ is a bilinear function of its arguments. This function is denoted $\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ and is called the *determinant* of the matrix $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$.

Space for your solution:

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = v \wedge w = \left(a_{1,1} \cdot \overrightarrow{i} + a_{2,1} \cdot \overrightarrow{j}\right) \wedge \left(a_{1,2} \cdot \overrightarrow{i} + a_{2,2} \cdot \overrightarrow{j}\right)$$

$$= \boxed{\text{bilinearity of the wedge product}} =$$

$$a_{1,1} \cdot a_{1,2} \cdot \overrightarrow{i} \wedge \overrightarrow{i} + a_{1,1} \cdot a_{2,2} \cdot \overrightarrow{i} \wedge \overrightarrow{j} + a_{2,1} \cdot a_{1,2} \cdot \overrightarrow{j} \wedge \overrightarrow{i} + a_{2,1} \cdot a_{2,2} \cdot \overrightarrow{j} \wedge \overrightarrow{j}$$

$$= a_{1,1} \cdot a_{1,2} \cdot 0 + a_{1,1} \cdot a_{2,2} \cdot 1 + a_{2,1} \cdot a_{1,2} \cdot (-1) + a_{2,1} \cdot a_{2,2} \cdot 0 = a_{1,1} \cdot a_{2,2} - a_{2,1} \cdot a_{1,2}.$$

(3). The system of linear equations $\begin{cases} a_{1,1}\cdot x + a_{1,2}\cdot y = b_1 \\ a_{2,1}\cdot x + a_{2,2}\cdot y = b_2 \end{cases}$ can also be written in the matrix form as a single matrix equation $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Interpreting this equation as a question about finding the suitable coefficients for the linear combination

$$x \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} + y \cdot \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

use the meaning and definition of wedge product to find the general formula for the solutions x and y. (This formula is called the *Cramer's rule.*)

Space for your solution:

Take the equation

$$x \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} + y \cdot \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad (*)$$

Wedge product from the left with vector $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$ applied to both sides of equation * yields:

$$x \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \wedge \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} + y \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \wedge \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \Leftrightarrow$$

$$x \cdot 0 + y \cdot \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \det \begin{bmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{bmatrix} \quad \Leftrightarrow \quad y = \frac{\det \begin{bmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{bmatrix}}{\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}} = \frac{a_{1,1}\cdot b_2 - b_1 \cdot a_{2,1}}{a_{1,1}\cdot a_{2,2} - a_{1,2}\cdot a_{2,1}}.$$

Wedge product from the right with vector $\begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$ applied to both sides of equation * yields:

$$x \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \wedge \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} + y \cdot \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} \wedge \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \wedge \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} \quad \Leftrightarrow$$

$$x \cdot \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + y \cdot 0 = \det \begin{bmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{bmatrix} \quad \Leftrightarrow \quad x = \frac{\det \begin{bmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{bmatrix}}{\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}} = \frac{b_1\cdot a_{2,2} - a_{1,2} \cdot b_2}{a_{1,1}\cdot a_{2,2} - a_{1,2}\cdot a_{2,1}}.$$

We are assuming that the denominators are not zero; otherwise there are a few degenerate cases that need to be considered separately. (The particular choices on which side to multiply the above equation by these two vectors are not significant. Their only point is to make the answer look nicer.)

Problem 4. Consider two vectors v and w in a three dimensional space $\mathbb{R}^3$ with the standard orientation. Define the *cross product* of these two vectors, denoted $v \times w$, as

$$v \times w = \begin{cases} 0 : \text{if the vectors } v \text{ and } w \text{ are linearly dependent;} \\ |v \wedge w| \cdot n : \\ \quad \text{where } n \text{ is the vector of unit length,} \\ \quad \text{perpendicular to the plane generated by } v \text{ and } w, \\ \quad \text{and such that } v, w, n \text{ is a basis of } \mathbb{R}^3 \text{ defining its standard orientation,} \\ \quad \text{otherwise.} \end{cases}$$

Note that the length $|v \wedge w|$ in the definition above is the area of the parallelogramm generated by v and w which does not depend on the choice of orientation in the plane of v and w.

(1). Take $\mathbb{R}^3$ with the orientation defined by the standard basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$. Identify $\mathbb{R}^2$ with the plane in $\mathbb{R}^3$ generated by $\overrightarrow{i}$ and $\overrightarrow{j}$. Prove that for any v and w in $\mathbb{R}^2 \subseteq \mathbb{R}^3$,

$$v \times w = (v \wedge w) \cdot \overrightarrow{k}$$

Space for your solution:

By definition, the cross product has the same length as the absolute value of the wedge product (namely, the area of the parallelogram generated by v and w). Also by definition, the cross product is perpendicular to the plane of v and w. Since these two vectors are in the plane generated by $\overrightarrow{i}$ and $\overrightarrow{j}$, the direction of the cross product must be parallel to that of $\overrightarrow{k}$. Thus the only thing that needs to be checked is the direction.

The value of $v \wedge w$ is positive $\Leftrightarrow$ orientation of v and w defines the standard orientation of $\mathbb{R}^2$ $\Leftrightarrow$ $v, w, \overrightarrow{k}$ defines the standard orientation of $\mathbb{R}^3$ $\Leftrightarrow$ the vector n in the definition of cross product is $\overrightarrow{k}$ itself.

(2). If $\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ 0 \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$, compute the value of $v \times w$ in terms of the $a_{i,j}$.

Space for your solution:

Using the result of subproblem (1),

$$v \times w = (v \wedge w) \cdot \overrightarrow{k} = \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \overrightarrow{k}$$

(3). If $\begin{bmatrix} a_{1,1} \\ 0 \\ a_{3,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ 0 \\ a_{3,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$, compute the value of $v \times w$ in terms of the $a_{i,j}$.

Space for your solution:

Now we cannot use directly the result of subproblem (1) because the vectors v and w are in the plane of $\overrightarrow{i}, \overrightarrow{k}$. But we can move the standard basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ in such a way that the plane of $\overrightarrow{i}, \overrightarrow{j}$ will turn into the plane of $\overrightarrow{i}, \overrightarrow{k}$. The key observation is the fact that this move turns $\overrightarrow{k}$ into $-\overrightarrow{j}$. Thus

$$v \times w = (v \wedge w) \cdot \overrightarrow{k} = -\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \cdot \overrightarrow{j}$$

(4). Suppose that $\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$. Prove that

$$v \times w = \det \begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \cdot \overrightarrow{i} - \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \cdot \overrightarrow{j} + \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \overrightarrow{k}$$

Use (without proof) the fact that $v \times w$ is a bilinear function of its arguments.

Space for your solution:

Using the results of the previous two sub-problems, and a similar result for the case of vectors in the plane of $\overrightarrow{j}, \overrightarrow{k}$, we can conlcule that the two sides coincide when v and w belong to the same coordinage plane, in particular, when v and w are the standard basis vectors. Using this and the fact that both sides are bilinear functions of v and w, we can conlcule that the equation holds for any v and w.

Problem 5. Suppose an umbrella has the shape of the upper half sphere:

$$U = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \;\&\; z \geq 0\}$$

(1). Find a parameterization of the umbrella.

Space for your solution:

For example, we can take the disk $\{(u, v) : u^2 + v^2 \leq 1\}$ in the (u, v) plane $\mathbb{R}^2$ and map it into the (x, y, z) space $\mathbb{R}^3$ via the function

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{1 - u^2 - v^2} \end{cases}$$

(2). We will model vertical rain as the vector field $F = -C \cdot \overrightarrow{k}$, where C is a constant characterizing the intensity of the rain. Find the amount of water per second that the umbrella protects you from. Hint: use the Gauß divergence theorem to simplify the computation.

Space for your solution:

Consider the volume contained under the umbrella

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1 \;\&\; z \geq 0\}.$$

$$\int_{\partial V} F \cdot \,\mathtt{d}S = \boxed{\text{Gauß divergence theorem}} = \int_V (\nabla \cdot F) \,\mathtt{d}x\,\mathtt{d}y\,\mathtt{d}z =$$

$$= \boxed{\text{since there is no source of liquid inside of } V\text{, the divergence } \nabla \cdot F = 0} = 0.$$

On the other hand, since the boundary ∂V is made up of the umbrella U and the bottom disk $D = \{(x, y, z) : x^2 + y^2 \leq 1 \;\&\; z = 0\}$, we have that:

$$\int_{\partial V} F \cdot \,\mathtt{d}S = \boxed{\text{additivity of integral}} = \int_U F \cdot \,\mathtt{d}S + \int_D F \cdot \,\mathtt{d}S$$

Thus the amount we are looking for — that entering V through the umbrella's surface — is

$$-\int_U F \cdot \,\mathtt{d}S = \int_D F \cdot \,\mathtt{d}S = \boxed{\text{on } D\text{, the } \mathtt{d}S = -\overrightarrow{k}\,\mathtt{d}x\,\mathtt{d}y} = C \int_D \mathtt{d}x\,\mathtt{d}y = C \cdot \pi.$$