Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 203, CRN 95613
Calculus with Analytic Geometry III

Final Exam

Instructor:

Name: Alexander Kasiukov

Office: Health, Science and Education Center, Room 109

Phone: (631) 851-6484

Email: kasiuka@sunysuffolk.edu

Web Site: `http://www.kasiukov.com`

Please print the requested information in the spaces provided:

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Email: *include to receive the final grade via email* ONLY *if you are not getting email updates*

- *Notes and books are permitted on this exam.*
- *Graphing calculators, computers, cell phones and any communication-capable devices are prohibited. Their mere presence in the open (even without use) is a sufficient reason for an immediate dismissal from this exam with a failing grade.*
- *You will not receive full credit if there is no work shown, even if you have the right answer. Use back pages if necessary. Please don't attach additional pieces of paper: if you run out of space, please ask for another blank final.*

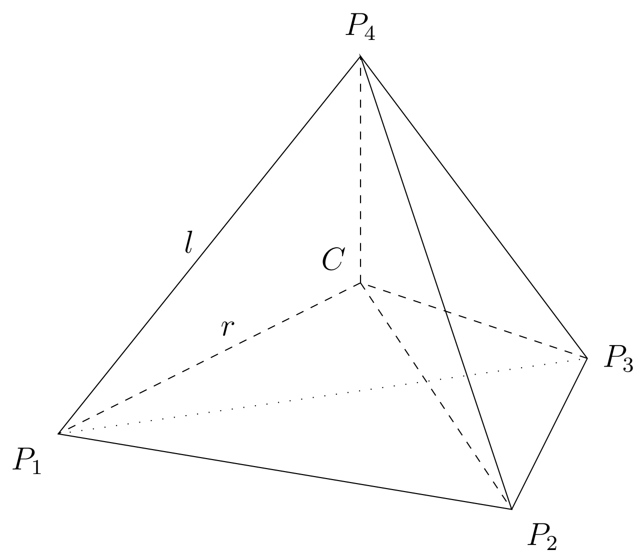
Problem 1. There are several lessons that may be drawn from solving this problem. Placing the situation at hand into a larger context may give additional tools for analyzing the situation. Considering the same situation in simpler circumstances (say, lower dimension) may give additional insight. A good choice of a system of coordinates is an essential solution step. A good system of coordinates should reflect the symmetries of the problem itself.

Suppose four points P_1, ... ,P_4 are positioned in a three dimensional space $\mathbb{R}^3$ in such a way that all the distances between them are equal: $\exists l \in \mathbb{R} : \forall i = 1 \ldots 4, \forall j = 1 \ldots 4 :$

$$i \neq j \;\Rightarrow\; |P_i P_j| = l.$$

In other words, these four points are the vertices of a regular tetrahedron and l is the length of its edge.

There exists unique fifth point $C \in \mathbb{R}^3$ equidistant (denote that distance r) from these four points. In other words, C is the center of the tetrahedron:



(1). Introduce a system of coordinates in a four dimensional space $\mathbb{R}^4$ in such a way that the coordinates of all the points P_1, ... ,P_4 are symmetric with respect to that system of coordinates. Hint: try to generalize from lower-dimensional cases. When you have two points P_1 and P_2 in a one dimensional space $\mathbb{R}$, you can place this $\mathbb{R}$ into $\mathbb{R}^2$ as the line $x+y=1$, so that $P_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $P_2 = \begin{bmatrix}0\\1\end{bmatrix}$, and $C = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$. When you have three points P_1, P_2 and P_3 in a two dimensional space $\mathbb{R}^2$, you can place this $\mathbb{R}^2$ into $\mathbb{R}^3$ as the plane $x+y+z=1$, so that $P_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $P_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $P_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, and $C = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$.
What are the coordinates of P_1, ... ,P_4 and C in this system of coordinates?

Space for your solution:

(2). Express the distance r from the point C to any of the points P_i in terms of the distance l between any of the two distinct points P_i, P_j. Hint: find two distances – between P_1 and P_2 and P_1 and C using coordinates introduced in the previous problem.

Space for your solution:

(3). Find the angle between $\overrightarrow{CP_i}$ and $\overrightarrow{CP_j}$ for $i, j = 1 \ldots 4, i \neq j$.

Space for your solution:

Problem 2.

Recall that the *orientation* of a space is defined by a choice of basis in that space. Two bases define the same orientation if and only if one of them can be continuously deformed into another while staying a basis. (More precisely, orientation is an equivalence class of bases, where the equivalence is defined as the possibility of deforming one basis into another continuously within the set of bases.) The special case of a zero-dimensional space necessitates separate definition. Orientation of a single point is the choice of coefficient $+1$ or -1 for that point. (There is also a more abstract way to define orientation that does not need to treat zero dimension as a special case.)

(1). Does the basis $\vec{i}, \vec{j}$ define the same orientation on $\mathbb{R}^2$ as $\vec{j}, \vec{i}$?

Space for your solution:

(2). Does the basis $\vec{i}, \vec{j}, \vec{k}$ define the same orientation on $\mathbb{R}^3$ as $\vec{j}, \vec{k}, \vec{i}$?

Space for your solution:

(3). How many orientations are there on an arbitrary space?

Space for your solution:

Problem 3.

Consider two vectors v and w on a plane $\mathbb{R}^2$ with the standard orientation. Define the *wedge product* of these two vectors, denoted $v \wedge w$, as

$$v \wedge w = \begin{cases} 0 : \text{if the vectors } v \text{ and } w \text{ are linearly dependent;} \\ (\text{area of the parrallelogram, generated by } v \text{ and } w) : \\ \quad \text{if } v, w \text{ is a basis of } \mathbb{R}^2 \text{ defining its standard orientation;} \\ -(\text{area of the parrallelogram, generated by } v \text{ and } w) : \\ \quad \text{if } v, w \text{ is a basis of } \mathbb{R}^2 \text{ defining the orientation, opposite to its standard one.} \end{cases}$$

(1). Determine the result of $v \wedge w$ when v and w are the standard basis vectors $\vec{i}$ and $\vec{j}$ in any combination.

Space for your solution:

(2). If $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\vec{i}, \vec{j}$, compute the value of $v \wedge w$ in terms of the $a_{i,j}$. Use (without proof) the fact that $v \wedge w$ is a bilinear function of its arguments. This function is denoted $\det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ and is called the *determinant* of the matrix $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$.

Space for your solution:

(3). The system of linear equations $\begin{cases} a_{1,1} \cdot x + a_{1,2} \cdot y = b_1 \\ a_{2,1} \cdot x + a_{2,2} \cdot y = b_2 \end{cases}$ can also be written in the matrix form as a single matrix equation $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Interpreting this equation as a question about finding the suitable coefficients for the linear combination

$$x \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} + y \cdot \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

use the meaning and definition of wedge product to find the general formula for the solutions x and y. (This formula is called the *Cramer's rule.*)

Space for your solution:

Problem 4. Consider two vectors v and w in a three dimensional space $\mathbb{R}^3$ with the standard orientation. Define the *cross product* of these two vectors, denoted $v \times w$, as

$$v \times w = \begin{cases} 0 : \text{if the vectors } v \text{ and } w \text{ are linearly dependent;} \\ |v \wedge w| \cdot n : \\ \quad \text{where } n \text{ is the vector of unit length,} \\ \quad \text{perpendicular to the plane generated by } v \text{ and } w, \\ \quad \text{and such that } v, w, n \text{ is a basis of } \mathbb{R}^3 \text{ defining its standard orientation,} \\ \quad \text{otherwise.} \end{cases}$$

Note that the length $|v \wedge w|$ in the definition above is the area of the parallelogramm generated by v and w which does not depend on the choice of orientation in the plane of v and w.

(1). Take $\mathbb{R}^3$ with the orientation defined by the standard basis $\vec{i}, \vec{j}, \vec{k}$. Identify $\mathbb{R}^2$ with the plane in $\mathbb{R}^3$ generated by $\vec{i}$ and $\vec{j}$. Prove that for any v and w in $\mathbb{R}^2 \subseteq \mathbb{R}^3$,

$$v \times w = (v \wedge w) \cdot \vec{k}$$

Space for your solution:

(2). If $\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ 0 \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\vec{i}, \vec{j}, \vec{k}$, compute the value of $v \times w$ in terms of the $a_{i,j}$.

Space for your solution:

(3). If $\begin{bmatrix} a_{1,1} \\ 0 \\ a_{3,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ 0 \\ a_{3,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\vec{i}, \vec{j}, \vec{k}$, compute the value of $v \times w$ in terms of the $a_{i,j}$.

Space for your solution:

(4). Suppose that $\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \end{bmatrix}$ and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{bmatrix}$ are the coordinates of v and w respectively in the standard basis $\vec{i}, \vec{j}, \vec{k}$. Prove that

$$v \times w = \det \begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \cdot \vec{i} - \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \cdot \vec{j} + \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \vec{k}$$

Use (without proof) the fact that $v \times w$ is a bilinear function of its arguments.

Space for your solution:

Problem 5. Suppose an umbrella has the shape of the upper half sphere:

$$U = \{(x,y,z) : x^2 + y^2 + z^2 = 1 \ \& \ z \geq 0\}$$

(1). Find a parameterization of the umbrella.

Space for your solution:

(2). We will model vertical rain as the vector field $F = -C \cdot \overrightarrow{k}$, where C is a constant characterizing the intensity of the rain. Find the amount of water per second that the umbrella protects you from. Hint: use the Gauß divergence theorem to simplify the computation.

Space for your solution: