MAT 142

Calculus with Analytic Geometry II

Spring 2025 Lecture Notes

An outline of Techniques of Integration

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Contents

1 Preliminaries

Remark (about *formatting of these notes*): The end of an item started with the **bold face font** is denoted by $\diamondsuit$

Remark (about *mathematical terms*): Mathematical terms will appear in *this font* when they are used for the first time in these notes. If the term in question is not immediately defined, it means that I assume you know its meaning. If this is not the case, please ask for an explanation. $\diamondsuit$

Remark (about *emphasis*): I will use *this font* to emphasize something. $\diamondsuit$

Any corrections and suggestions on how to improve these notes are very welcome.

Remark (about *mathematical notation*): Mathematics itself is the most rigorous of all fields of study, but the mathematical *notation* is rather sloppy. There is a simple explanation for that. Doing mathematics often involved writing down long computations. To focus on the important parts of those computations, all inessential details were usually suppressed in the notation. It is assumed that the intelligent reader of mathematics will be able to recover the implicit and the suppressed details based on their overall understanding of the subject from the context of the explicit part. However, when the context is to narrow to be implied, or is shifting in the course of considerations, traditional mathematical notation becomes ambiguous and may lead to errors in reasoning. Programming languages, on the other hand, are not optimized for brevity of handwritten expression, like mathematics was. They are meant to be *precise*, so that a non-intelligent computer could unambigously get their meaning. This is the reason programming languages and their notations may prove to be useful when we want to be precise. I will try to introduce new ideas in the most precise way possible, and then, once the concept is clarified with precise notation, I will show you the usual way it is denoted in mathematics. $\diamondsuit$

Definition (of `lambda abstraction`): Suppose X and Y are sets, $x \in X$, $f(x)$ is some expression that contains x, and $\forall x \in X : f(x) \in Y$. Then $\lambda x \in X.f(x) \in Y$ denotes the function with domain X, range Y and graph $\Big\{\Big(x, f(x)\Big) : x \in X\Big\}$. Often, when the sets X and Y can be implied from the context [1], they are ommitted and the function is denoted as $\lambda x.f(x)$. $\diamondsuit$

Example (functions of one real argument):

- the square function is $\lambda x.x^2$,
- the sin function is the same as $\lambda x.\sin(x)$,
- $f = \lambda x.x^2 + 3x + 2$ is usually written as $f(x) = x^2 + 3x + 2$.

$\diamondsuit$

Example (lambda notation making derivatives precise): When we are writing a derivative as $(x^2+3x)'$, we really mean $\Big(\lambda x.(x^2 + 3x)\Big)'$. Understood in its literal sense, the derivative $(x^2 + 3x)'$ would amount to that of a constant and result in zero. $\diamondsuit$

2 Theoretical basis of integration.

2.1 (Differential) 1-forms.

In Calculus I, we mostly considered functions of one real argument x. The domain and range of those functions were usually the set of all real numbers $\mathbb{R}$, or some subsets of that set. In Calculus II we shift our attention on 1-*forms* , whether we acknowledge that or not. I hope you appreciate the clarity this concept brings to the whole subject, albeit at a slight expense in the abstraction level.

Definition (of `linear space and vectors`): Given without proper foundation in *linear algebra* , this definition is somewhat "crippled". *Linear spaces* are, roughly speaking, sets of objects that can be added to each other, like $v+w$, and multiplied by a number C, as in $C\cdot v$, in a way that obeys the usual properties of addition and multiplication, like distributivity, commutativity, associativity etc. For example, we must have that $C\cdot(v+w) = C\cdot v + C\cdot w$. The objects of linear spaces are called *vectors* For our immediate needs, it is enough to substitue "linear space" with "set of real numbers" and "vector" with "real number". $\diamondsuit$

[1] The original lambda-calculus is type-free, so all inputs and all outputs are from the same set.

Definition (of `linearity of a function`): Suppose f is a function, and the domain and range of f are linear spaces. Then the function f is called *linear* if and only if:

- $\forall v, w \in \mathrm{Dom}\, f : f(v+w) = f(v) + f(w)$
- $\forall C \in \mathbb{R}, \forall v \in \mathrm{Dom}\, f : f(C \cdot v) = C \cdot f(v)$

When the domain and the range of f are simply $\mathbb{R}$, f being linear amounts to

$$f(x) = m \cdot x + b$$

with zero b. ♢

HOMEWORK: Prove the last statement.

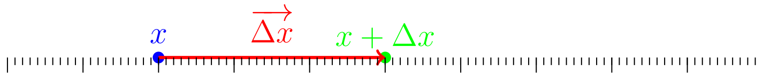
Definition (of `a 1-form`): A 1-form is a function of two vector arguments, linear with respect to one of them. ♢

Example (1-forms): Suppose $x, \Delta x \in \mathbb{R}$. Then the following functions (written using the λ-notation) are all 1-forms, linear with respect to the argument Δx:

- $\lambda(x, \Delta x).\ x \cdot \Delta x$,
- $\lambda(x, \Delta x).\ x^2 \cdot \Delta x$,
- $\lambda(x, \Delta x).\ \sin(x) \cdot \Delta x$.

♢

Remark (about *vector notation for the argument of a 1-form*): Given a 1-form $\lambda(x, \Delta x).\ \omega(x, \Delta x)$, think of x as a *point* on the real line and Δx as a *vector* starting at the point x and pointing right when Δx is positive and left when Δx is negative. This allows us to think of a 1-form as a function of one *fixed vector* argument $\lambda \overrightarrow{\Delta x}.\ \omega(\overrightarrow{\Delta x})$



Adopting this geometric idea, we can give meaning to the expression $\omega(\overrightarrow{AB})$, where A and B are some points on the real line, i.e. numbers:

$$\omega(\overrightarrow{AB}) = \omega(A, B - A)$$

This geometric point of view is very useful in the Calculus III course where we break the constraints of one dimensional line and consider things in multidimensional spaces. ♢

2.2 Differential of a function.

Given a function f that has linear spaces as its domain and range we define a 1-form related to that function. The process of obtaining such a 1-form is called *differentiation* and the resulting 1-form is called the *differential* of the function f and denoted $\mathtt{d}f$.

The following definition of differential works for functions defined on real number line $\mathbb{R}$. This case will satisfy our needs in this course and therefore this is the only definition that you really need to know for now.

Definition (of `differential of a single real argument function`): Suppose f is a function defined on an open interval of the real number line $\mathbb{R}$ and that interval contains x. Furthermore, assume that the range of f is a linear space, for example $\mathbb{R}$ itself. Then define the *differential* of the function as:

$$\mathtt{d}f(x, \Delta x) = f'(x) \cdot \Delta x,$$

where

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the *derivative* of f at the point x.

Thus, differential may or may not exist, depending on whether or not the derivative exists. If the derivative (and thus the differential) of the function exists on a given interval of real number line, the function is called *differentiable* on that interval ♢

Definition (of `differential of a function of a vector argument`): Suppose x is a vector, f is a function defined in a neighborhood of x, Δx is an arbitrary vector in the same linear space as x.

Then define $\mathtt{d}f(x, \Delta x)$ as a 1-form, linear with respect to Δx, such that

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x) - \mathtt{d}f(x, \Delta x)}{|\Delta x|} = 0,$$

where $|\Delta x|$ denotes the length of the vector Δx. If such a function exists, it is necessarily unique — hence we may say "the" differential. ♢

Remark (about *differential as linear approximation*): In practical terms, the definitions of differential mean that when Δx is close to zero, we have the approximate equality

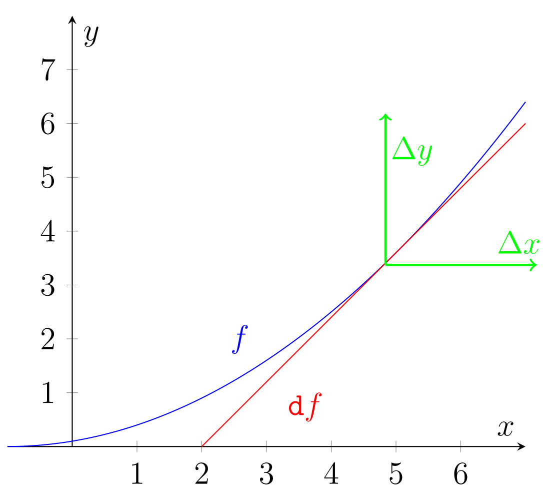
$$f(x + \Delta x) - f(x) \approx \mathtt{d}f(x, \Delta x),$$

or, in the geometric notation,

$$f(B) - f(A) \approx \mathtt{d}f(\overrightarrow{AB}).$$

The precision of this approximation increases as Δx becomes closer to zero. ♢

Remark (about *geometric meaning of differential*): Make the sketch of the graph of the function f in the usual (x, y)-coordinate system. Introduce a new $(\Delta x, \Delta y)$-coordinate system by shifting the origin of the (x, y)-coordinate system to the point $(x, f(x))$ in the old (x, y)-coordinate system. Then the graph of the differential $\mathtt{d}f$ as a function of Δx with fixed x in the $(\Delta x, \Delta y)$-coordinate system is the tangent to the graph of the function f. In particular, function being differentiable means geometrically that its graph has tangent, i.e. is smooth. ♢



Remark (about *short hand notation for differentials*): Sometimes we want to use a short hand notation for a function, without writing down its arguments explicitly: $\sqrt{\ }$ instead of $\sqrt{x}$, sin instead of $\sin(x)$ and the like. Since a differential of a function has two variables, it might be confusing to drop both. It is customary to drop reference to Δx and write $\mathtt{d}f(x)$ instead of $\mathtt{d}f(x, \Delta x)$.

So, for example, when one works with the function $f(x) = \sin(x)$, one might write $\mathtt{d}\sin(x)$ instead of $\mathtt{d}\sin(x, \Delta x)$. However, if we try to do the same to the right side of

$$\mathtt{d}\sin(x, \Delta x) = \cos(x) \cdot \Delta x,$$

we get complete nonsense:

$$\mathtt{d}\sin(x) = \cos(x)$$

We have a 1-form on the left hand side being equal to a usual function on the right hand side!

To deal with this situation, we will write $\mathtt{d}x$ in the place of Δx:

$$\mathtt{d}\sin(x) = \cos(x)\ \ \mathtt{d}x$$

It makes sense because, as we have seen above, if $f(x) = x$, then

$$\mathtt{d}f(x, \Delta x) = \Delta x$$

Therefore if we plug in Δx into $\mathtt{d}\sin(x) = \cos(x)\ \ \mathtt{d}x$, we get

$$\mathtt{d}\sin(x, \Delta x) = \cos(x)\ \ \mathtt{d}x(\Delta x) = \cos(x)\Delta x$$

— just what we want. $\diamondsuit$

2.3 Indefinite integral of a 1-form.

Definition (of `indefinite integral of a 1-form`): Suppose $\omega(x, \Delta x)$ is a 1-form, defined for any x in the open interval (a, b) and for any real Δx. By definition, $\int \omega$ is the set of all functions F, defined and *continuous* on the open interval (a, b), such that $\mathtt{d}F(x,\ \Delta x) = \omega(x,\ \Delta x)$ for all but finitely many $x \in (a, b)$ and all $\Delta x \in \mathbb{R}$:

$$\int \omega = \Big\{F(x) : F \text{ is continuous on the interval } (a,b) \bigwedge \forall_{\text{but finitely many}} x \in (a,b), \forall \Delta x \in \mathbb{R} :\ \mathtt{d}F(x,\ \Delta x) = \omega(x,\ \Delta x)\Big\}$$

$\diamondsuit$

Definition (of `antiderivative of a function`): Suppose f is a function, defined on an interval of the real line. By definition, F is the antiderivative of f, if and only if

1. F is a continuous function on that interval;
2. for all but finitely many x in that interval, $F'(x) = f(x)$.

In other words, F is an antiderivative of f if and only if F is an element of the set $\int f(x)\ \mathtt{d}x$. $\diamondsuit$

Suppose we are given a 1-form ω. We may ask the following questions:

1. Find at least one continuous function F, that has ω as its differential: $\omega(x) = \mathtt{d}F(x)$.
2. Find all possible continuous functions F, that have ω as their differential, i.e. find $\int \omega$.

It looks like the second question is harder to answer than the first one. However, it turns out that these questions have exactly the same level of difficulty. If you can find one function F, such that $\omega = \mathtt{d}F$ (answer question 1), then you can find all of them (answer question 2) by taking all the functions of the form $F(x) + C$, where C is an arbitrary constant [2]).

[2] We assume that the 1-form ω is defined on an interval of the real line. Sometimes we will deal with 1-forms (example: $\frac{\mathtt{d}z}{z}$) that are defined on a disjoint union of two or more intervals. For those 1-forms we must deal with each connected piece of the domain separately from others — as if we had several 1-forms, completely unrelated to each other.

HOMEWORK: Prove the preceding assertion: If F and G are functions defined and continuous on an interval of the real line, such that $\mathtt{d}f(x,\ \Delta x) = \mathtt{d}g(x,\ \Delta x)$ for all but finitely many x in that interval and for every Δx, then there exists a constant C, such that for every x in the interval, we have that $F(x) = G(x) + C$.

We can write it symbolically as $\int \omega = F(x) + C$. It really means that $\int \omega = \{F(x) + C : C \in \mathbb{R}\}$ [3]). The bottom line is: to recognize a 1-form ω as a differential of a function F is almost the same as to integrate ω.

Example (indefinite integration): Suppose we have the 1-form $\omega(x) = \cos(x)\ \ \mathtt{d}x$. We can find a function, for example $\sin(x)$, such that $\mathtt{d}\sin(x) = \cos(x)\ \ \mathtt{d}x$. Therefore $\int \cos(x)\ \ \mathtt{d}x = \sin(x) + C$. ♢

2.4 Definite integral of a 1-form over an interval.

Definition (of `definite integral of a 1-form`): Suppose we are given an interval $[a,\ b]$ and a 1-form $\omega(x,\ \Delta x)$, defined for every x in that interval.

Pick any integer n and divide the interval $[a,\ b]$ into n small intervals $[A_i, A_{i+1}]$, $i = 0, \ldots, n-1$, where $A_0 = a$ and $A_n = b$. Form the sum

$$\sum_{i=0}^{n} \omega(\overrightarrow{A_i A_{i+1}}).$$

(It is called *the Riemann sum* of the 1-form ω on the interval $[a,\ b]$, corresponding to the subdivision $A_0, A_1, \ldots, A_n$.)

By definition, the definite integral of a 1-form is the limit of its Riemann sums, achieved by taking finer and finer subdivisions of the interval of integration:

$$\int_{[a,\ b]} \omega = \lim_{\max_i |A_i A_{i+1}| \to 0} \sum_{i=0}^{n} \omega(\overrightarrow{A_i A_{i+1}}).$$

♢

[3] If the domain of F consists of several disjoint intervals, then the constants should be chosen independently for those intervals.

If the interval $[a,\ b]$ and the 1-form ω are fixed, then the definite integral $\int_{[a,\ b]}\omega$ (when it exists) is just a real number. If the integral exists, the 1-form in question is called *integrable* .

Theorem (*a sufficient condition of integrability of a* 1*-form*). *If the function $f(x)$ is continuous on a closed interval of the real line, then the* 1*-form $f(x)\ \mathtt{d}x$ is integrable over that interval.* ♢

2.5 The connection between the definite and indefinite integrals of a given 1-form.

It seems at first that the indefinite integral and definite integral have absolutely nothing in common. Indeed, given a 1-form ω, the indefinite integral of ω is a set of functions, whereas the definite integral of ω (over a fixed interval of integration) is just a number.

The connection is given by the following theorem.

Theorem (*the Newton-Leibnitz formula*). *Suppose $[a,\ b]$ is an interval of the real line and ω is a* 1*-form, integrable on the interval $[a,\ b]$. Then $\int_{[a,\ b]}\omega = F(x)\Big|_{[a,\ b]} = F(b) - F(a)$, where F is any function from the indefinite integral of ω. In particular, if F is a differentiable function, then $\int_{[a,\ b]}\ \mathtt{d}F(x) = F(x)\Big|_{[a,\ b]} = F(b) - F(a)$.* ♢

This theorem allows us to compute the definite integral, provided we know how to find the indefinite integral. **Example** (using Newton-Leibnitz formula to compute definite integral):

$$\int_{[0,\frac{\pi}{2}]} \cos(x)\ \ \mathtt{d}x =$$

$$= \boxed{\text{since } \cos(x)\ \ \mathtt{d}x =\ \mathtt{d}\ \sin(x)} =$$

$$= \int_{[0,\frac{\pi}{2}]} \mathtt{d}\ \sin(x) =$$

$$= \boxed{\text{using the Newton-Leibnitz formula}} =$$

$$= \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1.$$

$\diamondsuit$

3 Techniques of integration.

3.1 Obvious integrals, obtained by using the table of the derivatives.

The following three statements express exactly the same relationship between the functions $f(x)$ and $F(x)$:

- $f(x) = F'(x)$, i.e. the function $f(x)$ is the derivative of the function $F(x)$;
- $f(x)\ \mathtt{d}x = \mathtt{d}F(x)$, i.e. the differential of the function $F(x)$ is the 1-form $f(x)\ \mathtt{d}x$;
- $\int f(x)\ \mathtt{d}x = F(x) + C$ [4]).

Therefore every derivative that we know gives us an integral. It is important to remember the following derivatives (or integrals). (Okay, I will not torture you for not knowing the hyperbolic functions by heart.) For convenience, I will group them.

Power	
$F(x)$	$f(x)$
x^n	$n\ x^{n-1}$

Exponential	
$F(x)$	$f(x)$
a^x	$a^x \ln a$

Logarithmic	
$F(x)$	$f(x)$
$\log_a(x)$	$\frac{1}{x \ln a}$

[4]Again, assuming F and f are defined on one interval. If the domain of F and f consists of several disjoint intervals, use independent C for different intervals.

Trigonometric	
$F(x)$	$f(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x) = \frac{1}{\cos^2(x)}$
$\cot(x)$	$-\csc^2(x) = -\frac{1}{\sin^2(x)}$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

Inverse trigonometric	
$F(x)$	$f(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$
$\operatorname{arcsec}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arccsc}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Hyperbolic	
$F(x)$	$f(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$1-\tanh^2(x) = \operatorname{sech}^2(x)$
$\coth(x)$	$1-\coth^2(x) = -\operatorname{csch}^2(x)$
$\operatorname{sech}(x)$	$-\tanh(x)\operatorname{sech}(x)$
$\operatorname{csch}(x)$	$-\coth(x)\operatorname{csch}(x)$

Inverse hyperbolic	
$F(x)$	$f(x)$
$\operatorname{arcsinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$-\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$-\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

(By the way, what is the inverse of the power function?)

Example (the integrals "in" the above table):

$$\int \frac{1}{\sqrt{x^2+1}} \quad \mathtt{d}x = \operatorname{arcsinh}(x) + C$$

$$\int n\ x^{n-1} \quad \mathtt{d}x = x^n + C$$

$$\int a^x \ln(a) \quad \mathtt{d}x = a^x + C$$

$$\int \frac{1}{x \ln a} \quad \mathtt{d}x = \log_a(x) + C$$

♢

3.2 The basic properties of integrals: Linearity, Additivity, Integration by parts and Substitutions.

All of the fundamental properties of integrals (except additivity) come directly from properties of differentials.

The following comes from linearity of differentials:

Theorem (*linearity of integral*).

1. *If C is a constant and ω is an integrable 1-form, then $\int C\cdot\omega = C\cdot\int\omega$.*
2. *If ω and τ are integrable 1-forms, then $\int(\omega+\tau) = \left(\int\omega\right)+\left(\int\tau\right)$.*

The above is also true for definite integrals taken over the same fixed interval.
$\diamondsuit$

Example (using the linearity of integration):

1.
$$\int a^x \ \mathtt{d}x =$$
$$= \boxed{\text{Multiply by } 1 = \tfrac{1}{\ln(a)}\cdot\ln(a) \text{ to get the table integral of } a^x\ln(a).} =$$
$$= \int a^x \frac{1}{\ln(a)}\cdot\ln(a) \ \mathtt{d}x =$$
$$= \boxed{\text{Use the linearity of integration:}} =$$
$$= \frac{1}{\ln(a)}\int a^x\ln(a) \ \mathtt{d}x =$$
$$= \boxed{\text{Use the table integral}} =$$
$$= \frac{a^x}{\ln(a)} + C.$$

2.
$$\int x^m \ \mathtt{d}x =$$

$$= \boxed{\text{Take } n \text{ to be } m+1. \text{ Then } m = n-1.} =$$

$$= \int x^{n-1} \, \mathtt{d}x =$$

$$= \boxed{\text{If } n \neq 0, \text{ multiply by } \tfrac{1}{n} \cdot n \text{ to get the table integral of } n\, x^{n-1}} =$$

$$= \int \frac{1}{n} \cdot n \cdot x^{n-1} \, \mathtt{d}x =$$

$$= \boxed{\text{Using the linearity of integration:}} =$$

$$= \frac{1}{n} \cdot \int n \cdot x^{n-1} \, \mathtt{d}x =$$

$$= \boxed{\text{Using the table:}} =$$

$$= \frac{1}{n} \cdot \left(x^n + C\right) =$$

$$= \boxed{\text{Going back to the variable } m} =$$

$$= \frac{x^{m+1}}{m+1} + C.$$

The above integration works [5]) for all m except $m = -1$. To find the integral of $\frac{1}{x}$, take a closer look at the table integral

$$\int \frac{1}{x \ln a} \, \mathtt{d}x = \log_a(x) + C.$$

[5]If m is negative, then the 1-form $x^m \, \mathtt{d}x$ is not defined for $x = 0$. Therefore we can be sure that $x^m \, \mathtt{d}x$ is integrable on an interval $[a,\ b]$ (and use the Newton-Leibnitz formula to find that integral) only if the interval under consideration does not contain 0. To be completely precise, in the case of $m < -1$ we should write

$$\int x^m \, \mathtt{d}x = \left\{ \frac{x^{m+1}}{m+1} + C(x) : \ C(x) \text{ is a constant for } x > 0 \text{ and } x < 0 \right\},$$

in other words the integral consists of all the functions of the form

$$f(x) = \begin{cases} \frac{x^{m+1}}{m+1} + C_1, & \text{if } x > 0 \\ \frac{x^{m+1}}{m+1} + C_2, & \text{if } x < 0 \end{cases},$$

where C_1 and C_2 are arbitrary (and independent from each other!) constants.

When $a = e$ it becomes

$$\int \frac{1}{x} \ \mathtt{d}x = \ln(x) + C.$$

(This makes sense only for positive x.)

♢

Theorem (*additivity of definite integral*).

1. *If ω is a 1-form and a, b are real numbers, then $\int_{[a,\ b]} \omega = -\int_{[b,a]} \omega$.*
2. *If ω is a 1-form and a, b, c are real numbers, then $\int_{[a,\ c]} \omega = \left(\int_{[a,\ b]} \omega\right) + \left(\int_{[b,\ c]} \omega\right)$.*

♢

The following comes from the differential of the product formula:
Theorem (*integration by parts*). *If f and g are differentiable functions, then*

$$\int f(x) \ \mathtt{d}g(x) = f(x)g(x) - \int g(x) \ \mathtt{d}f(x)$$

The above is also true for definite integrals taken over the same fixed interval:

$$\int_{[a,\ b]} f(x) \ \mathtt{d}g(x) = f(x)g(x)\Big|_{[a,\ b]} - \int_{[a,\ b]} g(x) \ \mathtt{d}f(x) =$$

$$f(b)g(b) - f(a)g(a) - \int_{[a,\ b]} g(x) \ \mathtt{d}f(x)$$

♢

The following comes from the chain rule for differentials:
Theorem (*substitution*). *If f is a continuous function and g is a differentiable function, then*

$$\int f\big(g(x)\big) \ \mathtt{d}g(x) = \left(\int f(u) \ \mathtt{d}u\right)\Big|_{u=g(x)}.$$

If in addition a, b are real numbers, then

$$\int_{[a,\ b]} f\big(g(x)\big) \ \mathtt{d}g(x) = \int_{[g(a),\ g(b)]} f(u) \ \mathtt{d}u.$$

$\diamondsuit$

Remark (about *two uses of substitution*): The above theorem may be used in two different ways.

1. We are given the integral $\int f\big(g(x)\big) \ \mathtt{d}g(x) = \int f\big(g(x)\big)g'(x) \ \mathtt{d}x$. We go to the integral $\int f(u) \ \mathtt{d}u$. After computing $\int f(u) \ \mathtt{d}u$, we substitute $u = g(x)$ into the answer. This is called the *direct substitution* .
   Example (using direct substitution):

$$\int \frac{\sin(\ln x)}{x} \ \mathtt{d}x =$$

$$= \boxed{\text{group the denominator and the } \mathtt{d}x} =$$

$$\int \sin(\ln x) \frac{\mathtt{d}x}{x} =$$

$$= \boxed{\text{since } \tfrac{\mathtt{d}x}{x} = \mathtt{d}\ln x} =$$

$$\int \sin(\ln x) \ \mathtt{d}\ln x =$$

$$= \boxed{\text{using } g(x) = \ln x} =$$

$$\int \sin(u) \ \mathtt{d}u =$$

$$= \boxed{\text{using the table of integrals}} =$$

$$-\cos(u) + C =$$

$$= \boxed{\text{substituting } u = g(x)} =$$

$$-\cos(\ln x) + C.$$

$\diamondsuit$

2. We are given the integral $\int f(u) \ \mathtt{d}u$ and we are introducing a substitution $u = g(x)$, that takes us to the integral $\int f\big(g(x)\big)g'(x) \ \mathtt{d}x$. After computing $\int f\big(g(x)\big)g'(x) \ \mathtt{d}x$, we have to *invert* the function g and substitute $x = g^{-1}(u)$ into the answer. This is called the *inverse substitution* . **Example** (using inverse substitution):

$$\int \frac{\sin(\ln x)}{x} \ \mathtt{d}x$$

$= \boxed{\text{introduce the substitution } x = e^t} =$

$$\int \frac{\sin(\ln e^t)}{e^t} \ \mathtt{d}e^t$$

$= \boxed{\text{find } \mathtt{d}e^t} =$

$$\int \frac{\sin(\ln e^t)}{e^t} \ e^t \ \mathtt{d}t$$

$= \boxed{\text{simplify}} =$

$$\int \sin(t) \ \mathtt{d}t$$

$= \boxed{\text{use the table of integrals}} =$

$$-\cos(t) + C$$

$= \boxed{\text{substitute the inverse of } x = e^t\text{, which is } t = \ln x} =$

$$-\cos(\ln x) + C.$$

♢

♢

Example (using substitution to compute "the other half of" $\int \frac{1}{x} \ \mathtt{d}x$): For positive x, we found that

$$\int \frac{1}{x} \ \mathtt{d}x = \ln(x) + C.$$

To find the integral of $\frac{1}{x}$ for negative x, we can use the technique of substitution and the fact that the function $\lambda(x \in \mathbb{R} - 0) : \ \frac{1}{x}$ is *odd* .

HOMEWORK: Prove that

1. one of the antiderivatives of an *even* function is an odd function;
2. every antiderivative of an odd function is an even function.

Assume that $x < 0$. Than we have:

$$\int \frac{1}{x} \; \mathtt{d}x = \int -\frac{1}{-x} \; \mathtt{d}x = \int \frac{1}{-x} \; \mathtt{d}(-x)$$

= Using the direct substitution $u = -x$ and noticing that $u > 0$: =

$$\int \frac{1}{u} \; \mathtt{d}u = \ln u + C$$

= Going back to the variable x: =

$$\ln(-x) + C.$$

$\diamondsuit$

Remark (about *general formula for* $\int \frac{1}{x} \; \mathtt{d}x$): The two formulas:

$$\int \frac{1}{x} \; \mathtt{d}x = \begin{cases} \ln(x) + C : \text{if the domain of } \frac{1}{x} \text{ is assumed to be } (0, +\infty) \\ \ln(-x) + C : \text{if the domain of } \frac{1}{x} \text{ is assumed to be } (-\infty, 0) \end{cases}$$

may be combined into one formula

$$\int \frac{1}{x} \; \mathtt{d}x = \ln|x| + C$$

that works for any non-zero x [6]). $\diamondsuit$

Remark (about *linear adjustment of substitution*): The (direct) substitution technique can be immediately applied when we have an integral of the

[6] With the understanding that the additive constant C should be chosen independently for $x > 0$ and $x < 0$.

form $\int f\big(g(x)\big) \ \ \mathtt{d}g(x) = \int f\big(g(x)\big)g'(x) \ \ \mathtt{d}x$. We can extend this area of applicability to the case of $\int f\big(g(x)\big)h(x) \ \ \mathtt{d}x$, where h differs from g' by a multiplicative constant. The reason is quite simple: linearity of both integral and differential allows us to multiply the function under the integral by whatever constant, as long as we compensate for it. ♢

Example (linear adjustment of substitution):

$$\int \sin(3x-5) \ \ \mathtt{d}x =$$

$=\boxed{\text{we could immediately apply substitution } u = 3x-5 \text{ if we had } \int \sin(3x-5)\cdot 3 \ \ \mathtt{d}x}$
$=$

$$\frac{1}{3}\int \sin(3x-5) \ \ \mathtt{d}(3x-5) =$$

$=\boxed{\text{use the substitution } u = 3x-5}=$

$$\frac{1}{3}\int \sin(u) \ \ \mathtt{d}u = -\frac{\cos(u)}{3} + C$$

$=\boxed{\text{go back to } x}=$

$$-\frac{\cos(3x-5)}{3} + C.$$

♢

3.3 Specific substitution for specific situations.

Unlike the process of differentiation, integration is often a highly non-obvious task. Because of that, we have to consider different classes of functions and develop appropriate techniques for those classes.

However, all of the following rules of integration are, to some degree, tentative. By no means will they give the best way of integration in every case they are used.

Note that by now, in addition to the table functions (i.e. those that appear as $f(x)$ in the table on the page 11), we can integrate the following:

1. Any power of x.

HOMEWORK: Prove that the formula $\int x^n \ \mathtt{d}x = \frac{x^{n+1}}{n+1} + C$ works for any *real* number $n \neq -1$, for instance $n = \pi, \sqrt{2}$ etc.

2. Any polynomial in x. (Use the linearity of integral to reduce the problem to that of integrating a power of x.)

3. Any exponent in x, like e^{ax+b}. (Use the linear substitution $u = ax + b$ to get a table function.)

4. Any polynomial in sines and cosines, multiplied by any exponent as in the preceding case. (Use the exponential formula for sines and cosines to make this into a sum of exponent and then the linearity of integral.) This is very hard in practice, therefore we will do something different, when we can.

The list of topics presented here is by no means exhaustive.

3.3.1 Rational functions.

Definition (of `rational functions`): Rational function is a function of the form

$$\frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not identically zero. $\diamondsuit$

The process of integrating such functions is straightforward, but sometimes very tedious. Suppose we are given a rational function

$$\frac{p(x)}{q(x)}.$$

It can be integrated by doing the following steps:

1. Perform *long division* of $p(x)$ by $q(x)$ to obtain the expression

$$\frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)},$$

where $f(x)$ is the polynomial resulting in such division, $r(x)$ is the remainder of such division, and $\deg\Big(r(x)\Big) < \deg\Big(q(x)\Big)$.

2. Factor completely the polynomial $q(x)$:

$$q(x) = {p_1}^{d_1}(x) \cdot \ldots \cdot {p_n}^{d_n}(x),$$

where the $p_i(x)$ are distinct *irreducible* polynomials. (This is potentially the hardest step in the whole process. We can not effectively factor a polynomial of degree higher than 4.)

Remark (about *irreducible polynomials*): If we consider polynomials with complex coefficients, then the only irreducible polynomials are those of degree 1, i.e. the polynomials $ax + b$ with $a \neq 0$.

If we consider polynomials with real coefficients only, the irreducible polynomials are

(a) polynomials of degree 1;

(b) polynomials of degree 2 that have negative *discriminant* [7].

$\diamondsuit$

3. Find the *partial fraction* decomposition of $\frac{r(x)}{q(x)}$. It is the sum of the form

$$\frac{r(x)}{q(x)} = \sum_{i=1\ldots n} \sum_{j=1\ldots d_i} \frac{f_{i,j}(x)}{{p_i}^j(x)},$$

where $f_{i,j}(x)$ are polynomials having $\deg\Big(f_{i,j}(x)\Big) < \deg\Big(p_i(x)\Big)$.

4. At this point we have to integrate the sum of following types of functions.

- A polynomial $f(x)$. Can be done (see above).
- A fraction of the type $\frac{c}{(ax+b)^n}$, where a, b, c are real and n is a positive integer constants. Using the substitution $u = ax + b$ we will get an integral of power of u.

[7] Recall that the discriminant of the quadratic polynomial $ax^2 + bx + c$ is the expression $b^2 - 4ac$.

- A fraction of the type $\frac{ax+b}{(Ax^2+Bx+C)^n}$, where a, b, A, B, C are real and n is a positive integer constants. Complete the square to get the fraction in the form $\frac{ax+b}{(x^2+C)^n}$ (possibly with different a and b). Integrate the fraction $\frac{ax}{(x^2+C)^n}$ using substitution $u = x^2 + C$. We still have to find the integral of $\frac{b}{(x^2+C)^n}$. Using linearity of integral, it can be made into $h_n(x) = \frac{1}{(x^2+1)^n}$. One possible way to integrate it is induction on n, using integration by parts. Here is how it goes. First,

$$\int \frac{1}{x^2+1} \ \mathtt{d}x = \arctan x + C.$$

Second,

$$\int \frac{1}{(x^2+1)^n} \ \mathtt{d}x =$$

$$= \boxed{\text{replace 1 with } x^2+1-x^2 \text{ and use the linearity of integral}} =$$

$$\int \left(\frac{x^2+1}{(x^2+1)^n} - \frac{x^2}{(x^2+1)^n}\right) \ \mathtt{d}x =$$

$$\int \frac{1}{(x^2+1)^{n-1}} \ \mathtt{d}x - \frac{1}{2}\int x \cdot \frac{\mathtt{d}(x^2+1)}{(x^2+1)^n} =$$

$$= \boxed{\text{prepare the second integral for integration by parts}} =$$

$$\int \frac{1}{(x^2+1)^{n-1}} \ \mathtt{d}x + \frac{1}{2(n-1)}\int x \cdot \ \mathtt{d}\left(\frac{1}{(x^2+1)^{n-1}}\right) =$$

$$= \boxed{\text{integrate the second integral by parts}} =$$

$$\int \frac{1}{(x^2+1)^{n-1}} \ \mathtt{d}x + \frac{x}{2(n-1)} \cdot \left(\frac{1}{(x^2+1)^{n-1}}\right) - \frac{1}{2(n-1)}\int \left(\frac{1}{(x^2+1)^{n-1}}\right) \ \mathtt{d}x =$$

$$= \boxed{\text{combine the like terms (find the lowest common denominator etc.)}}$$

$$= \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{1}{(x^2+1)^{n-1}} \ \mathtt{d}x.$$

This gives an inductive formula for the integral

$$\int \frac{1}{(x^2+1)^n} \ \mathtt{d}x.$$

Another way is to use the substitution $x = C \ \tan(t)$.

3.3.2 Trigonometric functions.

There are several cases that we can handle.

1. $$\int \sin^m(x)\ \cos^n(x)\ \mathtt{d}x,$$

   where m and n are integer numbers (including the case $m = 0$ or $n = 0$). Consider subcases:

   (a) One of the numbers m, n is odd. Use the corresponding function as the direct substitution.

   (b) Both numbers are even non-negative. Use the double angle formulas to lower the degrees:

   $$\sin^2(x) = \frac{1 - \cos(2x)}{2},$$
   $$\cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

2. $$\int \sin(mx)\ \cos(nx)\ \mathtt{d}x,$$
   $$\int \cos(mx)\ \cos(nx)\ \mathtt{d}x,$$
   $$\int \sin(mx)\ \sin(nx)\ \mathtt{d}x,$$

   where m and n are natural numbers

3. General rational function of trigonometric functions (all trigonometric functions are applied to just x, unlike the preceding case). Use the substitution $u = \tan(\frac{x}{2})$. Then

   $$\mathtt{d}x = \frac{2\ \mathtt{d}u}{1 + u^2},$$
   $$\sin(x) = \frac{2u}{1 + u^2},$$
   $$\cos(x) = \frac{1 - u^2}{1 + u^2}.$$

3.3.3 Linear function under a radical.

If the integral contains $\sqrt[m]{ax+b}$, where m is integer, a and b are real constants, then the substitution $u = \sqrt[m]{ax+b}$ will rationalize the radical.

3.3.4 Square root of a quadratic function.

1. Complete the square.
2. Use a linear substitution to get a radical in standard form.
3. Find your radical in the standard form in the "$\sqrt{\ }$" column of the following table (assuming $a > 0$).
4. Use one of the corresponding substitutions on the right to get rid of the radical.
5. Use the corresponding inverse function to get back from t to x.

See the remarks after the table for the explanation of the $\pm$.

Square root of a quadratic function				
$\sqrt{\ }$	$x =$	$t \in$	$\sqrt{\ } =$	$dx =$
$\sqrt{a^2 - x^2}$	$a\ \sin(t)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$a\ \cos(t)$	$a\ \cos(t)\ dt$
	$a\ \cos(t)$	$[0, \pi]$	$a\ \sin(t)$	$-a\ \sin(t)\ dt$
$\sqrt{a^2 + x^2}$	$a\ \tan(t)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$a\ \sec(t)$	$a\ \sec^2(t)$
	$a\ \sinh(t)$	$(-\infty, \infty)$	$a\ \cosh(t)$	$a\ \cosh(t)$
$\sqrt{x^2 - a^2}$	$\pm\ a\ \sec(t)$	$[0, \frac{\pi}{2})$	$a\ \tan(t)$	$\pm\ a\ \sec(t)\tan(t)\ dt$
	$\pm\ a\ \cosh(t)$	$[0, \infty)$	$a\ \sinh(t)$	$\pm\ a\ \sinh(t)\ dt$

The case of $\sqrt{\ } = \sqrt{x^2 - a^2}$ deserves a special consideration. The graph of the function $\sqrt{x^2 - a^2}$ consists of two disjoint halves of hyperbolas. The domain (i.e. possible values of x) is the disjoint union of two intervals

$$(-\infty,\ -a] \cup [a,\ +\infty).$$

Each of those intervals should be considered separately. This explains the need for $\pm$ in front of the two substitutions for that radical.

References

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