Suffolk County Community College

**Michael J. Grant Campus**

# Department of Mathematics

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# MAT 141
# Calculus with Analytic Geometry I

## Final Exam: Solutions and Answers

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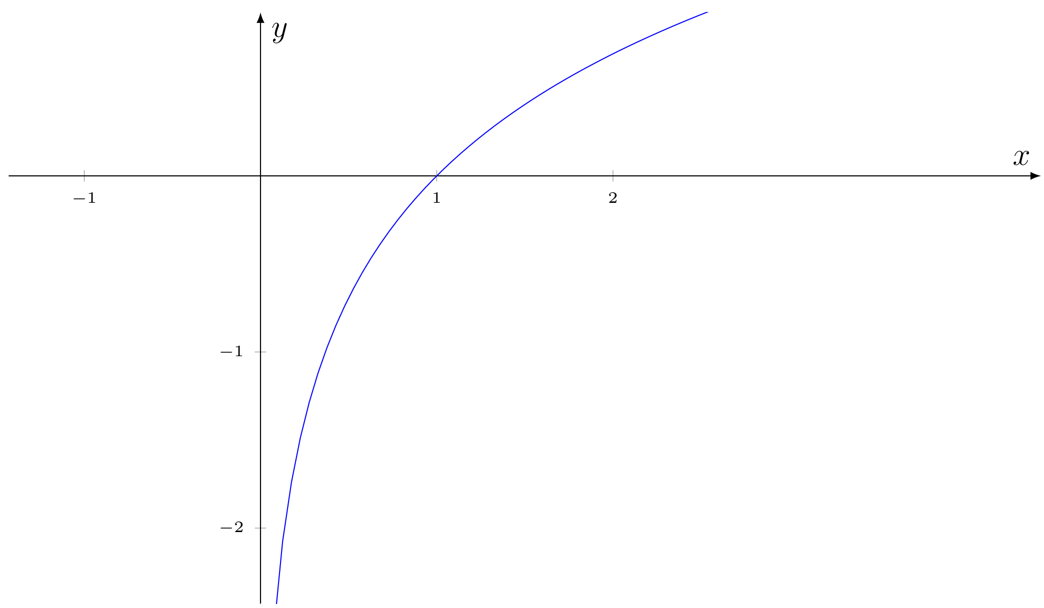
**Problem 1.** Consider the following claim:

$$\lim_{x\to 0+}\Big(\ln(x)\Big) = -\infty.$$

**(1).** Make a sketch of the graph of the function ln and determine, based on the graph, if the above claim is true.

*Space for your solution:*

As $x$ approaches 0 from the right, the value of $\ln(x)$ does indeed approach $-\infty$, thus making it intuitively clear that the statement about limit is correct.




**(2).** Using the definition of limit, translate the above claim into a statement about inequalities.

*Space for your solution:*

Applying the general definition $\left(\lim_{x\to x_0+}\Big(f(x)\Big) = -\infty\right) \quad \Leftrightarrow$

$$\Bigg(\forall M > 0 \left(\exists \delta > 0 \left(\forall x \left(x_0 < x < x_0 + \delta \Rightarrow f(x) < -M\right)\right)\right)\Bigg)$$

to our situation, we get:

$$\Bigg(\forall M > 0 \left(\exists \delta > 0 \left(\forall x \left(0 < x < \delta \Rightarrow \ln(x) < -M\right)\right)\right)\Bigg).$$

**(3).** Based on the definition of limit, determine if the claim in question is true. (Call $M$ the variable that controls closeness of $\ln(x)$ to $-\infty$ and $\delta$ — the one that controls the closeness of $x$ to 0. Demonstrate the truth of the limit-claiming statement by providing an explicit value of $\delta$ for any given $M$, or the falsehood — by producing a counterexample value of $M$.)

*Space for your solution:*

Take the last inequality from the definition of the limit:

$$\ln(x) < -M \Leftarrow\!\boxed{\text{apply exponent to both sides}}\!\Rightarrow e^{\ln(x)} < e^{-M} \Leftrightarrow$$

$$\Leftarrow\!\boxed{\text{cancel exponent and logarithm, remembering about domain of the latter}}\!\Rightarrow$$

$$\Leftrightarrow \begin{cases} x < e^{-M} \\ x > 0 \end{cases} \Leftrightarrow 0 < x < e^{-M}.$$

Thus: $\left(\lim\limits_{x\to 0+}\Big(\ln(x)\Big) = -\infty\right) \Leftarrow\!\boxed{\text{solution of the previous sub-problem}}\!\Rightarrow$

$$\Leftrightarrow \Bigg(\forall M > 0 \Bigg(\exists \delta > 0 \Bigg(\forall x \Big(0 < x < \delta \Rightarrow \ln(x) < -M\Big)\Bigg)\Bigg)\Bigg) \Leftrightarrow$$

$$\Leftarrow\!\boxed{\text{the above inequality solution}}\!\Rightarrow$$

$$\Leftrightarrow \Bigg(\forall M > 0 \Bigg(\exists \delta > 0 \Bigg(\forall x \Big(0 < x < \delta \Rightarrow 0 < x < e^{-M}\Big)\Bigg)\Bigg)\Bigg) \Leftrightarrow$$

$$\Leftrightarrow \Bigg(\forall M > 0 \Bigg(\exists \delta > 0 \Big((0, \delta) \subseteq (0, e^{-M})\Big)\Bigg)\Bigg) \Leftrightarrow$$

$$\Leftarrow\!\boxed{\text{the interval } (0,\delta) \subseteq (0, e^{-M}) \text{ iff } \delta \le e^{-M}}\!\Rightarrow$$

$$\Leftrightarrow \Bigg(\forall M > 0 \Bigg(\exists \delta > 0 \Big(\delta \le e^{-M}\Big)\Bigg)\Bigg) \Leftrightarrow$$

$$\Leftarrow\!\boxed{\text{taking the } \delta = e^{-M} \text{ as the explicit witness for the existential quantifier}}\!\Rightarrow$$

$$\Leftrightarrow \Bigg(\forall M > 0 \Bigg(\exists \delta = e^{-M}\Bigg)\Bigg) \Leftrightarrow \text{TRUE.}$$

**Problem 2.** Use the limit of composition theorem and cancellation of the main terms (based on the properties of logarithms) to find

$$\lim_{x\to 0+}\left(\ln\left(10x^2\right)-\ln\left(3x\right)\right).$$

*Space for your solution:*

Direct application of the limit of composition gives:

$$\lim_{x\to 0+}\left(\ln\left(10x^2\right)-\ln\left(3x\right)\right)=\left(\lim_{x\to 0+}\left(\ln\left(10x^2\right)\right)\right)-\left(\lim_{x\to 0+}\left(\ln\left(3x\right)\right)\right)=$$
$$=(-\infty)-(-\infty),$$

which is an indeterminacy. Therefore, finding the limit requires cancellation of the main terms of each $\infty$. This can be accomplished by utilizing properties of logarithms:

$$\lim_{x\to 0+}\left(\ln\left(10x^2\right)-\ln\left(3x\right)\right)=\boxed{\text{logarithm of fraction formula}}=\lim_{x\to 0+}\left(\ln\left(\frac{10x^2}{3x}\right)\right)=$$
$$=\lim_{x\to 0+}\left(\ln\left(\frac{10}{3}x\right)\right)=\left(\lim_{x\to 0+}\left(\ln\left(\frac{10}{3}\right)\right)\right)+\left(\lim_{x\to 0+}\left(\ln\left(x\right)\right)\right)=$$
$$=\boxed{\text{value of the limit from the first problem}}=\ln\left(\tfrac{10}{3}\right)+\left(-\infty\right)=-\infty.$$

**Problem 3.** Consider the function with range $\mathbb{R}$ defined as $f(x)=\ln(x)$ on the maximum set of real numbers $x$ for which this formula makes sense.

**(1).** What is the domain of the function $f$?

*Space for your solution:*

The formula that defines $f$ makes sense for all positive values of $x$, therefore

$$\texttt{Dom}\ f=\{x\in\mathbb{R}:x>0\}=(0,+\infty).$$

**(2).** Is the function $f(x)$ continuous at $x = 5$? Why? (No proof is needed, but please state the meaning of continuity in terms of limits.)

*Space for your solution:*

Yes, because $\lim_{x \to 5} \ln(x) = \ln(5)$.

**(3).** Is the function $f$ continuous?

*Space for your solution:*

Yes. A function is continuous if it is continuous at every point in its domain. Similar to subproblem (2), the equality $\lim_{x \to x_0} f(x) = f(x_0)$ holds for every positive value of $x_0$, meaning that $f$ is continuous at every $x_0$ in its domain $(0, +\infty)$.

**(4).** Suppose that

- the domain $(0, +\infty)$ of the function $f$ is compactified by adding 0 and $+\infty$, and
- the range $\mathbb{R}$ of $f$ is compactified by adding $\{-\infty, +\infty\}$.

Can the function $f$ be extended to become a continuous function from $[0, +\infty]$ to $\mathbb{R} \cup \{-\infty, +\infty\}$? If so, provide the extention; if not, give the reason why it is impossible.

*Space for your solution:*

Yes: the new function $F$ defined as

$$F(x) = \begin{cases} f(x), & \text{if } x \in (0, +\infty) \\ -\infty, & \text{if } x = 0 \\ +\infty, & \text{if } x = +\infty \end{cases}$$

is continuous on the whole $[0, +\infty]$. The continuity of this extension follows from

$$\lim_{x \to 0+} \Big( \ln(x) \Big) = -\infty \qquad \text{and} \qquad \lim_{x \to +\infty} \Big( \ln(x) \Big) = +\infty.$$

**Problem 4.** Find $\frac{\mathtt{d}\sin\left(\frac{x^2}{\cos(x)}\right)}{\mathtt{d}x}$. (The answer should not contain any operations of derivative, but other than that does not have to be simplified in any way.)

*Space for your solution:*

$$\frac{\mathtt{d}\sin\left(\frac{x^2}{\cos(x)}\right)}{\mathtt{d}x} = \boxed{\text{chain rule}} = \frac{\mathtt{d}\sin\left(\frac{x^2}{\cos(x)}\right)}{\mathtt{d}x\left(\frac{x^2}{\cos(x)}\right)} \cdot \frac{\mathtt{d}\left(\frac{x^2}{\cos(x)}\right)}{\mathtt{d}x}$$

$$= \boxed{\text{derivative of sin}} = \cos\left(\frac{x^2}{\cos(x)}\right) \cdot \frac{\mathtt{d}\left(\frac{x^2}{\cos(x)}\right)}{\mathtt{d}x}$$

$$= \boxed{\text{derivative of a fraction}} = \cos\left(\frac{x^2}{\cos(x)}\right) \cdot \frac{\frac{\mathtt{d}x^2}{\mathtt{d}x}\cdot\cos(x) - x^2 \cdot \frac{\mathtt{d}\cos(x)}{\mathtt{d}x}}{\cos^2(x)}$$

$$= \boxed{\text{derivative of a power and cos}} = \cos\left(\frac{x^2}{\cos(x)}\right) \cdot \frac{2x\cos(x) - x^2\left(-\sin(x)\right)}{\cos^2(x)}$$

$$= \boxed{\text{algebra}} = \cos\left(\frac{x^2}{\cos(x)}\right) \cdot \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)}.$$

**Problem 5.** In this problem we will study the extrema and the points of extrema of the function $f(x) = 4x - 3x^3$ on the interval $[0, 2]$.

**(1).** Identify the suspect points of extrema.

*Space for your solution:*

The internal points of the domain where the derivative of the function is not zero cannot be extremum points. The suspects are those points where one of these two conditions are violated. Thus, we need to consider

- the end points of the domain, $x = 0, 2$;
- the points where the derivative is undefined, (none such points exist for the function under consideration); and
- the points where the derivative is defined and is zero: $0 = f'(x) = (4x - 3x^3)' = 4 - 9x^2$ $\Leftrightarrow$ $4 = 9x^2$ $\Leftrightarrow$ $x = \frac{2}{3}$ (omitting the root $x = -\frac{2}{3}$ which is outside of the domain of the function $f$.)

So, the suspect points of extrema are $\left\{0, \frac{2}{3}, 2\right\}$.

**(2).** Find the points of extrema and the extrema of the function $f$.

*Space for your solution:*

Evaluating the function $f$ in all suspect points of extrema $\left\{0, \frac{2}{3}, 2\right\}$, we get the following table:

| $x$ | $f(x)$ |
|---|---|
| $0$ | $4 \cdot 0 - 3 \cdot 0^3 = 0$ |
| $\frac{2}{3}$ | $4 \cdot \frac{2}{3} - 3 \cdot (\frac{2}{3})^3 = \frac{8}{3} - \frac{24}{27} = \frac{72}{27} - \frac{24}{27} = \frac{48}{27} = \frac{16}{9}$ |
| $2$ | $4 \cdot 2 - 3 \cdot 2^3 = 8 - 24 = -16$ |

demonstrating that

- the maximum of $f$ is $\frac{16}{9}$ and the point of maximum is $\frac{2}{3}$;
- the minimum of $f$ is $-16$ and the point of minimum is $2$.

**Problem 6.** The arresting gear system of an aircraft carrier enables landing of high-speed aircraft within limited distance. It includes an arresting wire that is layed across the deck. The wire is grasped and pulled out by the aircraft's tailhook. The arresting wire transmits tension to hydraulic arresting gears, which dissipate the kinetic energy of the aircraft by hydraulic damping.

We will use a very simplistic model of hydraulic damping, assuming that the damping mechanism works as an elastic spring with force $F(x) = H \cdot x$, where $F$ is the damping force (in Newtons), $x$ is the displacement (in meters), and $H$ is the Young's modulus (in $\frac{kg}{m \cdot s^2}$ or $\frac{N}{m}$). In this problem we will derive the Young's modulus $H$ of material that is needed to stop an aircraft landing on a deck of an aircraft carrier.

**(1).** Find the formula, in terms of $H$ and $L$, for the work $W = \int_0^L F(x)\, dx$ against the damping force $F(x)$ done when moving an aircraft along the deck for $L$ meters from the relaxed position of the arrester wire.

*Space for your solution:*

$$W = \int_0^L F(x)\, dx = \int_0^L H \cdot x \cdot dx = H \cdot \int_0^L x\, dx = H \cdot \int_0^L d\left(\frac{x^2}{2}\right)$$
$$= H \cdot \left(\frac{x^2}{2}\right)\Big|_{x=0}^{L} = H \cdot \left(\left(\frac{L^2}{2}\right) - \left(\frac{0^2}{2}\right)\right) = \frac{HL^2}{2}.$$

**(2).** As the aircraft moves along the deck, it uses up its kinetic energy $E = \frac{M \cdot v^2}{2}$ to do the work against the dumping force. In order for the aircraft's speed to be reduced to zero at the end of landing, the initial kinetic energy $E$ of the aircraft must equal to the total amount of work $W$ done by the arresting gear system. Using the formula for $W$ from above, express the Young's modulus $H$ sufficient for stopping an aircraft of mass $M$ with the landing speed $v$ over the landing deck of length $L$.

*Space for your solution:*

$W = E \Leftarrow$ $\boxed{\text{use the expression for } W \text{ from the previous problem}}$ $\Rightarrow$

$$\frac{HL^2}{2} = \frac{M \cdot v^2}{2} \quad \Leftrightarrow \quad H = \frac{Mv^2}{L^2}.$$

**(3).** Assume that the mass of the F-18 Hornet at landing is 18,000 kg, its landing speed is 60 $\frac{\mathrm{m}}{\mathrm{s}}$, and the landing runway is 200 m long. Compute the Young's modulus sufficient for these conditions.

*Space for your solution:*

$$H = \frac{Mv^2}{L^2} = \frac{18,000\mathrm{kg} \cdot (60\ \frac{\mathrm{m}}{\mathrm{s}})^2}{(200 \cdot \mathrm{m})^2} = 1,620\ \frac{\mathrm{kg}}{\mathrm{s}^2} = 1,620\ \frac{\mathrm{N}}{\mathrm{m}}.$$