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MTH 150

Calculus I

August 25, 2023

Contents

1 Lecture 0 – Number Systems

The presentation here mostly follows *Mathematical Analysis I* by V. Zorich: [**Zorich2015**] and [**Zorich2016**]. See also [**Silverman1969**].

1.1 Real Numbers $\mathbb{R}$

DEFINITION 1. A set $\mathbb{R}$ is called the *set of real numbers* and its elements are called *real numbers* if the following list of conditions holds, called the axiom system of the real numbers.

1.1.1 I. Field Axioms

DEFINITION 2. A *binary operation* on a set X is a function f from $X \times X$ to X. Usually instead of $f(a, b)$ we write $a \circ b$. Addition and multiplication (of integers or rational numbers) are examples of binary operations.

DEFINITION 3. A set $\mathbb{F}$ with two binary operations on it written as "addition" ("$+$") and "multiplication" (using the usual symbol "$\cdot$" or simply juxtaposition) is called a *field* if it satisfies the following axioms:

I.1 Addition is *associative*, i.e. for all $x, y, z \in \mathbb{F}$,

$$(x + y) + z = x + (y + z).$$

I.2 There is a special element in $\mathbb{F}$ called *additive identity* or just *zero* written as 0 with the property

$$x + 0 = 0 + x = x$$

for all $x \in \mathbb{F}$.

I.3 For every $x \in \mathbb{F}$ there is a special element in $\mathbb{F}$ called *additive inverse* or *opposite* written $-x$ with the property

$$x + (-x) = (-x) + x = 0.$$

I.4 Addition is *commutative*, i.e. for all $x, y \in \mathbb{F}$,

$$x + y = y + x.$$

I.5 Multiplication is *associative*, i.e. for all $x, y, z \in \mathbb{F}$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

I.6 There is a special element in $\mathbb{F}$ called *multiplicative identity* (different from zero!) written as 1 with the property

$$x \cdot 1 = 1 \cdot x = x$$

for all x in $\mathbb{F}$.

I.7 For any $x \neq 0$ in $\mathbb{F}$ there exists an element in $\mathbb{F}$ called the *multiplicative inverse* or *reciprocal* of x written as x^{-1} for which

$$xx^{-1} = x^{-1}x = 1.$$

I.8 Multiplication is *commutative*, i.e. for any x and y in $\mathbb{F}$,

$$x \cdot y = y \cdot x.$$

I.9 There holds the distributive law: for all $x, y, z \in \mathbb{F}$ we have

$$x \cdot (y + z) = xy + xz.$$

1.1.2 II. Order Axioms

The symbol $\wedge$ means "and", and the $\vee$ means "or".

Between elements of $\mathbb{R}$ there is a *relation* $\leqslant$, that is, for elements $x, y \in \mathbb{R}$ one can determine whether $x \leqslant y$ or not. Here the following conditions must hold:

II.1 The relation is *reflexive*, i.e. for every $x \in \mathbb{R}$, $x \leqslant x$.

II.2 For any $x, y \in \mathbb{R}$, $x \leqslant y$ and $y \leqslant x$ imply $x = y$.

II.3 The relation is *transitive*, i.e. for every $x, y, z \in \mathbb{R}$ we have

$$(x \leqslant y) \wedge (y \leqslant z) \Longrightarrow x \leqslant z.$$

II.4 For any $x, y \in \mathbb{R}$ we have $(x \leqslant y) \vee (y \leqslant x)$.

The relation $\leqslant$ on $\mathbb{R}$ is called *inequality*.

A set satisfying axioms II.1, II.2, and II.3 is called *partially ordered*. If, in addition, the set satisfies, axiom II.4, the set is called *linearly ordered* or *totally ordered*.

1.1.3 III. The Connection between addition and order on $\mathbb{R}$

For any $x, y, z \in \mathbb{R}$ we have

$$(x \leqslant y) \Longrightarrow (x + z \leqslant y + z).$$

1.1.4 IV. The Connection between multiplication and order on $\mathbb{R}$

For any $x, y \in \mathbb{R}$ we have

$$\Big((0 \leqslant x) \wedge (0 \leqslant y)\Big) \Longrightarrow (0 \leqslant xy).$$

1.1.5 V. The Axiom of Completeness (Continuity)

If X and Y are nonempty subsets of $\mathbb{R}$ having the property that $x \leqslant y$ for all $x \in X$ and all $y \in Y$, then there exists $c \in \mathbb{R}$ such that $x \leqslant c \leqslant y$ for all $x \in X$ and all $y \in Y$.

Note that the set $\mathbb{Q}$ of rational numbers introduced below doesn't have this property. See Problem 6.

1.2 Consequences of the Axioms of Real Numbers

Theorem 1.

1. There is only zero in $\mathbb{R}$.
2. Each element of $\mathbb{R}$ has a unique negative.
3. The equation $a + x = b$, where $a, b \in \mathbb{R}$ has a unique solution $x = b + (-a)$ written also as $x = b - a$.
4. There is only one multiplicative unit in the real numbers.
5. Every element of $\mathbb{R}$ other than zero has only one multiplicative reciprocal.
6. For any nonzero a in $\mathbb{R}$ and any b in $\mathbb{R}$, the equation $ax = b$ has a unique solution $x = a^{-1}b$, also written as $x = b/a$.
7. For any x in $\mathbb{R}$, $x \cdot 0 = 0 \cdot x = 0$.
8. For any x, y in $\mathbb{R}$, $x \cdot y = 0$ implies that $x = 0$ or $y = 0$.
9. For any x in $\mathbb{R}$, $-x = (-1) \cdot x$.
10. For any x in $\mathbb{R}$, $(-1)(-x) = x$.
11. For any x in $\mathbb{R}$, $(-x)(-x) = x \cdot x$.

We begin by noting that the relation $x \leqslant y$ (read "x is less than or equal to y") can also be written as $y \geqslant x$ ("y is greater than or equal to x"); when $x \neq y$, the relation $x \leqslant y$ is written $x < y$ (read "x is less than y") or $y > x$ (read "y is greater than x"), and is called *strict inequality*.

Theorem 2.

1. For any x and y in $\mathbb{R}$ precisely one of the following relations hold: $x < y$, $x = y$, or $x > y$.
2. For any x, y, z in $\mathbb{R}$ we have
$$(x < y) \wedge (y \leqslant z) \implies (x < z)$$
$$(x \leqslant y) \wedge (y < z) \implies (x < z)$$
3. For any x, y, z, w in $\mathbb{R}$
$$(x < y) \implies (x + z) < (y + z),$$
$$(0 < x) \implies (-x < 0),$$
$$(x \leqslant y) \wedge (z \leqslant w) \implies (x + z) \leqslant (y + w)$$
$$(x \leqslant y) \wedge (z < w) \implies (x + z) < (y + w).$$
4. If x, y, z are in $\mathbb{R}$ then
$$(0 < x) \wedge (0 < y) \implies (0 < xy),$$
$$(x < 0) \wedge (y < 0) \implies (0 < xy),$$
$$(x < 0) \wedge (0 < y) \implies (xy < 0),$$
$$(x < y) \wedge (0 < z) \implies (xz < yz),$$
$$(x < y) \wedge (z < 0) \implies (yz < xz).$$
5. $0 < 1$.
6. $(0 < x) \implies (0 < x^{-1})$ and $(0 < x) \wedge (x < y) \implies (0 < y^{-1}) \wedge (y^{-1} < x^{-1})$.

DEFINITION 4. A set $X \subseteq \mathbb{R}$ is said to be *bounded from above* (resp. *bounded from below*) if there exists a number c in $\mathbb{R}$ such that $x \leqslant c$ (resp. $c \leqslant x$) for all x in X. The number c in this case is called an *upper bound* (resp. *lower bound*) of the set X. A set that is bounded both from above and from below is called *bounded*.

DEFINITION 5. An element a in X is called the *largest* or *maximal* (resp. *smallest* or *minimal*) element of X if $x \leqslant a$ (resp. $a \leqslant x$) for all x in X.

NOTE 1. If X has a maximal (minimal) element, it is necessarily unique (*think which axiom guarantees that!*). However not every set has a maximal (minimal) element.

DEFINITION 6 (Supremum/Infinum)**.** The smallest number that bounds a set $X \subseteq \mathbb{R}$ from above is called the *least upper bound* (or the *exact upper bound*) of X and denoted $\sup X$ (read "the supremum of X"). The largest number that bounds a set $X \subseteq \mathbb{R}$ from below is called the *greatest lower bound* (or the *exact lower bound*) of X and denoted $\inf X$ (read "the infinum of X").

DEFINITION 7 (Supremum/Infinum)**.** The number $\alpha \in \mathbb{R}$ is called the *least upper bound* (or the *exact upper bound*) of X and denoted $\sup X$ (read "the supremum of X") if 1) for all $x \in X$, $x \leqslant \alpha$, and 2) for any $\varepsilon > 0$ there exists $x \in X$ such that $x > \alpha - \varepsilon$. Similarly the number β is called the *greatest lower bound* (or the *exact lower bound*) of X and denoted $\inf X$ (read "the infinum of X") of 1) for all $x \in X$, $\beta \leqslant x$, and 2) for any $\varepsilon > 0$ there exists $x \in X$ such that $x < \beta + \varepsilon$.

The two definitions are in fact equivalent. See Problem 7.

Theorem 3 (Least Upper Bound Principle)**.** Every nonempty set of real numbers that is bounded from above has a unique least upper bound.

Of course a similar result is true about a set bounded from below having a unique greatest lower bound.

Theorem 4 (Triangle Inequality)**.** For any a, b in $\mathbb{R}$,

$$|a+b| \leqslant |a| + |b|,$$
$$|a-b| \geqslant \Big||a| - |b|\Big|.$$

1.3 Rational Numbers $\mathbb{Q}$

1.3.1 A Definition

We are familiar with the set of *integers* $\mathbb{Z}$ and the set of *natural numbers* (positive integers) $\mathbb{N}$. We would now like to introduce the set $\mathbb{Q}$ of *rational numbers*. Consider *fractions*, i.e. ratios of integers of the form

$$\frac{m}{n},$$

where $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$.

Here the symbol "$\in$" means "belongs to" or "is an element of ". The symbol "$\setminus$" refers to set-theoretic subtraction: for two sets A and B, the set $A \setminus B$ is the set of all the elements in A that are not in B.

Arithmetic operation on fractions are defined in a familiar way:

$$\begin{aligned} \frac{m}{n} \pm \frac{m'}{n'} &= \frac{mn' \pm m'n}{nn'}, \\ \frac{m}{n} \cdot \frac{m'}{n'} &= \frac{mm'}{nn'}. \\ \frac{\frac{m}{n}}{\frac{m'}{n'}} &= \frac{mn'}{m'n}, \end{aligned} \tag{1}$$

Note that the left-hand side of (1) only makes sense when $m' \neq 0$ and $n \neq 0$, which implies that the right-hand side of (1) makes sense.

Two fractions m/n and m'/n' are said to be *equal* if $mn' = m'n$. For example, the fractions

$$\frac{4}{-2}, \frac{-6}{3}, \frac{100}{-50}$$

are all equal and we can think of them as representing the same number, called a *rational number*. The set of all rational numbers will be denoted by $\mathbb{Q}$. Identifying any integer m with $m/1$ we see that the set $\mathbb{Z}$ of integers is a proper subset of $\mathbb{Q}$: $\mathbb{Z} \quad \mathbb{Q}$.

1.3.2 A More Formal Approach to Defining Rational Numbers

A more thorough definition of *rational numbers* requires the concept of *equivalence relation*. Recall that a *relation* R on a set A is any collection of pairs (x, y) with $x, y \in A$. We can then write xRy or $x \overset{R}{\sim} y$ or just $x \sim y$ if R is understood from the context. A relation on a set A is called an *equivalence relation* if

- $x \sim x$ for any $x \in A$ (*reflexivity property*);
- $x \sim y \Longrightarrow y \sim x$ for all $x, y \in A$ (*symmetry property*);
- $\Big(x \sim y\Big) \wedge \Big(y \sim z\Big) \Longrightarrow x \sim z$ (*transitivity property*).

(Here the symbol $\wedge$ means *and*, much like the symbol $\vee$ means *or*.) It is known that every equivalence relation on a given set A determines a *partition* of A, i.e. a collection of nonempty pairwise disjoint subsets of A whose union is all of A. This partition is naturally provided by *equivalence classes*, i.e. the sets

$$O_a = \{x \in A : x \sim a\}, \quad a \in A.$$

See and do problem 2.

Here is one more symbol to discuss: given two sets A and B we define their *Cartesian product*

$$A \times B := \{(a, b) : (a \in A) \wedge (b \in B)\},$$

i.e. the set of all (ordered) pairs in which the first coordinates are from the first set and the second coordinates are from the second set. Consider now an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ defined via

$$(m, n) \sim (m', n') \iff mn' = m'n.$$

We call each equivalence class under this equivalence relation a rational number. The arithmetic operation are formally defined as

$$(m, n) \pm (p, q) := (mq \pm pn, nq), \tag{2}$$
$$(m, n) \cdot (p, q) := (mp, nq), \tag{3}$$
$$\frac{(m, n)}{(p, q)} := (mq, np). \tag{4}$$

Note that much like in 1, the both the left-hand side and the right-hand side of 4 make sense as long as $n \neq 0$ and $p \neq 0$.

Note that all three definitions (2), (3), and (4) are actually operation defined on equivalence classes via particular representatives, and therefore we must verify that all three operations are *well-defined*, i.e. that their results do not depend on the choice of particular representatives. These verifications are routine and are left as an exercise (see Problem 3).

1.4 Archimedian Property of $\mathbb{R}$

1.5 Problems

1. Adopt the argument given in class to show that the numbers $\sqrt{3}$, $\sqrt{6}$, and $\sqrt{2}+\sqrt{3}$ are irrational.

2. A *partition* of a set A is a collection $\{A_i\}_{i\in I}$ of subsets of A satisfying the following three properties:

   (a) $A_i \neq \emptyset$ for any $i \in I$;

   (b) $A_i \neq A_j$ for any $i, j \in I$ provided that $i \neq j$;

   (c) $\bigcup_{i\in I} A_i = A$.

   Show that any partition of A defines an equivalence relation on A wherein $x \sim y \iff x, y \in A_i$ for some $i \in I$.

   Show conversely that given an equivalence relation on A, its equivalence classes $O_a = \{x \in A : x \sim a\}$, $a \in A$, form a partition of A.

3. Show that the operations on equivalence classes defining rational numbers in equations (2), (3), and (4) are *well-defined* (that is, they do not depend on the choice of representatives of those equivalence classes). For example, for (2) you are to prove that

$$(m, n) + (p, q) \sim (m', n') + (p', q')$$

   whenever $(m, n) \sim (m', n')$ and $(p, q) \sim (p', q')$.

4. Prove that

$$|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}| \leqslant |a-c| + |b-d|$$

   for any $a, b, c, d \in \mathbb{R}$.
   Hint. Use the identity

$$\sqrt{\alpha} - \sqrt{\beta} = \frac{\alpha - \beta}{\sqrt{\alpha}+\sqrt{\beta}}, \quad \alpha, \beta > 0.$$

5. Read about the *method of mathematical induction*. I recommend the following sources (all available on the Internet):

   - *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* by M. Bona.
   - *Abstract Algebra: An Introduction* by T. Hungerford.
   - *Modern Calculus and Analytic Geometry* by R. Silverman.

   Now do the following problems.

   (a) Prove that for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

   (b) Prove that for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (c) Prove that for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

   (d) Prove that for $x \in \mathbb{R}$ satisfying $x > -1$ and any $n \in \mathbb{N}$,

$$(1+x)^n \geqslant 1 + nx.$$

(e) Prove that for any $n \in \mathbb{N}$, $n \geqslant 3$, there holds $n^{n+1} > (n+1)^n$.

(f) Prove that the sum of the interior angles of an n-gon (a polygon with n sides) is $\pi \cdot (n-2)$.
Hint. Triangulate the n-gon.

(g) Prove that the plane is divided into precisely

$$\frac{n^2 + n + 2}{2}$$

parts by n straight lines, no two of which are parallel and no three of which intersect at one point.

(h) Given an equal arm balance capable of determining only the relative weights of two quantities and eight coins, all of equal weight except possibly one that is lighter, explain how to determine if there is a light coin and how to identify it in just two weighings.
Generalize this result: Given an equal arm balance and $3^n - 1$ coins, $n \geqslant 1$, all of equal weight except possibly one that is lighter, show how to determine if there is a light coin and how to identify it in at most n weighings.

6. Show that $\mathbb{Q}$ does not enjoy the completeness property, i.e. there exists nonempty subsets A and B of $\mathbb{Q}$ such that for all $a \in A$ and all $b \in B$, $a \leqslant b$, however there does not exist $c \in \mathbb{Q}$ such that for all $a \in A$ and all $b \in B$, $a \leqslant c \leqslant b$. This is a fundamental difference between $\mathbb{R}$ and $\mathbb{Q}$.

7. Verify that the two definitions (Definitions 6 and 7) are logically equivalent. That is, if α (resp. β) is supremum (resp. infinum) of a set X in the sense of one of these definitions then it is so in the sense of the other definition.

2 Lecture 1 – The Limit Concept

For the rest of the course we will denote by $\mathbb{R}$ the set of *real numbers* (I recommend that you work through the introductory lecture explaining the construction of real numbers); by $\mathbb{Q}$ the set of *rational numbers* (make sure you know what they are); by $\mathbb{Z}$ the set of *integers* (these are constructed from natural numbers); by $\mathbb{N}$ the set of *natural numbers* (according to L. Kronecker[1], "God created the natural numbers. All else is the work of man"). Finally, by $\mathbb{Z}_{\geqslant 0}$ we will denote the set of nonnegative integers.

We consider a real-valued function f defined on a subset D of $\mathbb{R}$.

2.1 Definition and Examples

DEFINITION 8 (Neighborhood/Deleted Neighborhood)**.** Given a point $a \in \mathbb{R}$, a *neighborhood* of a is any open interval containing a. A *deleted neighborhood* of a is a neighborhood of a with a removed. A δ*-neighborhood* of a, $\delta > 0$, is the interval $(a - \delta, a + \delta)$. A *deleted* δ*-neighborhood* of a is a δ-neighborhood of a with a removed, i.e. any set $(a - \delta, a) \cup (a, a + \delta)$.

Here the symbol $\in$ means "belongs to" or "is an element of" and refers to set membership. The notation $x \in A$ means that x is an element of a set A. Say, $\square \in \{Ⓢ, \triangle, \square, \star\}$ is a true statement.

The symbol $\cup$ means "union of two sets". When A and B are two sets, $A \cup B$ is the set whose elements are exactly the ones that belong to at least one of the sets A or B. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 5\}$ then $A \cup B = \{1, 2, 3, 5\}$.

NOTE 2. Any nonempty neighborhood contains a δ-neighborhood for some $\delta > 0$. Similarly, any nonempty deleted neighborhood contains a deleted δ-neighborhood for some $\delta > 0$.

DEFINITION 9 (Limit Point/Accumulation Point)**.** A point $a \in \mathbb{R}$ is called a *limit point* (or sometimes an *accumulation point*) of D if any *deleted* neighborhood of a contains at least one point of D.

NOTE 3. A limit point of a set may or may not be an element of the set.

NOTE 4. Verify that a point $a \in \mathbb{R}$ is an accumulation point of D if and only if every deleted neighborhood of a contains *infinitely many* (!) points of D.

[1] Leopold Kronecker (1823–1891) was a German mathematician known for his work in number theory, algebra, and logic.

Example 1 (Examples of Sets and their Limit Points)**.** For an open interval (a, b), $a < b$, every point of the closed interval $[a, b]$ is a limit point and there are no others.

The only limit point of the set $\left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$ is zero.

The set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$. *Think why!*

DEFINITION 10 (ε-δ definition of limit)**.** Let $a \in \mathbb{R}$ be a limit point of $D = \text{Dom}(f)$. We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches (or tends to) a"

or

"f has limit L as x tends to a",

and write

$$f(x) \to L \quad \text{as} \quad x \to a,$$

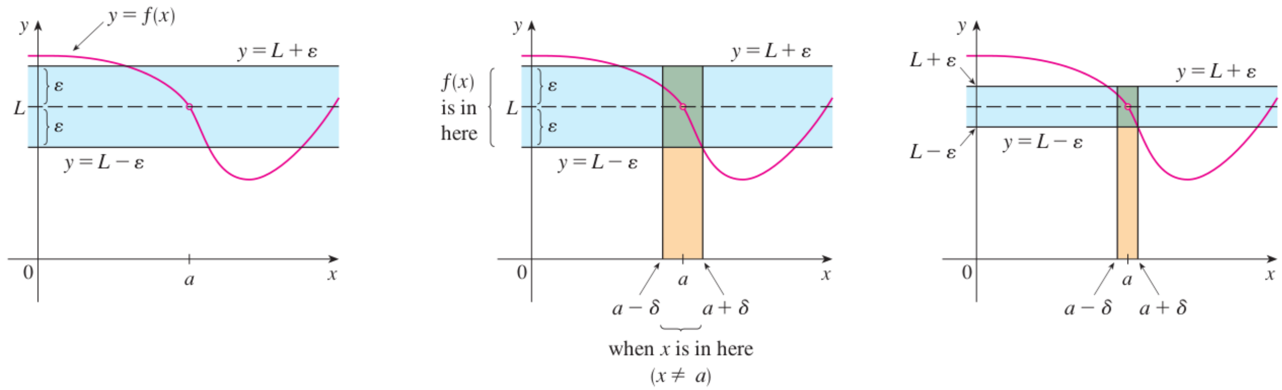
or

$$f(x) \xrightarrow[x \to a]{} L,$$

or

$$\lim_{x \to a} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < |x - a| < \delta$, there holds $|f(x) - L| < \varepsilon$.



In logical symbolism this definition is written as

$$\lim_{D \ni x \to a} f(x) = L := \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D \; \left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right).$$

Here the symbols $\forall$ and $\exists$ are, respectively, the *universal quantifier* and *existential quantifier*. The universal quantifier stands for "for all" or "for each" (in fact, in the LaTeX 2_ε typesetting system in which this document was typed, $\forall$ is typed as `\forall`.) The existential quantifier $\exists$ stands for "there exists…" or "there is…" (and by analogy is typed in LaTeX 2_ε as `\exists`).

NOTE 5. As a is a limit point of D, there necessarily exist points $x \in D$ satisfying the inequality $0 < |x - a| < \delta$ in Definition 26, so the definition is not *vacuous*.

NOTE 6. In Definition 26, δ is a function of both a and ε.

Example 2. For any $a \in \mathbb{R}$, $\lim_{x \to a} x = a$, and in the definition of limit one may take $\delta = \varepsilon$.

Example 3. For any $a \in \mathbb{R}$, $\lim_{x \to a} x^2 = a^2$.

Solution: Suppose we are given $\varepsilon > 0$, and consider $|f(x) - a^2| = |x^2 - a^2| = |x-a||x+a|$. Since $x \to a$, we may assume that x is at distance at most 1 from a, i.e. that $a-1 < x < a+1$. Then $2a-1 < x+a < 2a+1$, and thus $|x+a| < M := \max\{|2a-1|, |2a+1|\}$. Now if $\delta := \min\{\frac{\varepsilon}{M}, 1\}$, then

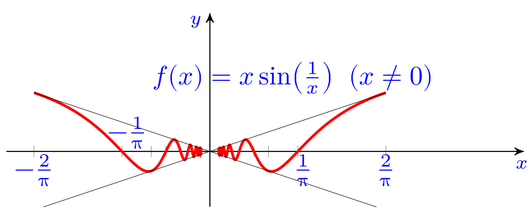
$$0 < |x-a| < \delta \quad \text{implies} \quad |x^2 - a^2| = |x-a||x+a| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Example 4. For any $a \in \mathbb{R}$, $a \geqslant 0$, $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$.

Solution: Suppose we are given $\varepsilon > 0$. If $a = 0$ then $\delta = \varepsilon^2$ works. If $a > 0$, consider $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}$. If $|x-a| < a$ so $x > 0$, we have $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{a}} < \varepsilon$ whenever $|x-a| < \varepsilon\sqrt{a}$. Thus we may take $\delta = \min\{a, \varepsilon\sqrt{a}\}$.

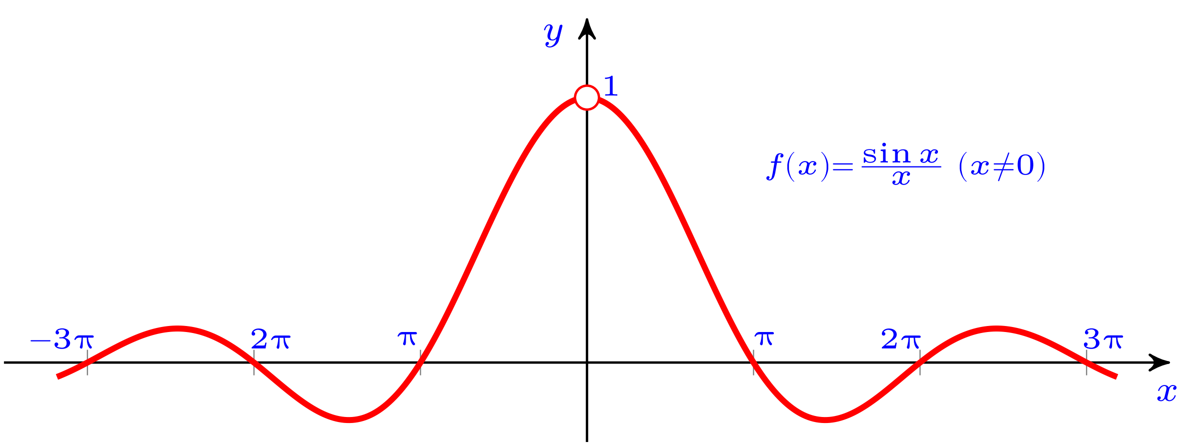
The symbol $\setminus$ in the context of sets means "set difference". When A and B are two sets, the set $A \setminus B$ consists precisely of those elements of A that are not elements of B. For example, if $A = \{1,2,3\}$, $B = \{1,3\}$ then $A \setminus B = \{2\}$.

Example 5. Let $D = \mathbb{R} \setminus \{0\}$ and let $f\colon D \to \mathbb{R}$ be given by $f(x) = x \sin\left(\frac{1}{x}\right)$. Verify that $\lim\limits_{x \to 0} f(x) = 0$, and in Definition 26 one can take $\delta = \varepsilon$.



NOTE 7. Example 5 shows that a function $f\colon D \to \mathbb{R}$ may have a limit as $x \to a$ without even being defined at the point a itself. This is exactly the situation that most often arises when limits must be computed. This circumstance is taken into account in our definition of limit, where we wrote the strict inequality $0 < |x-a|$.

Example 6. We will prove a very important result here: $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

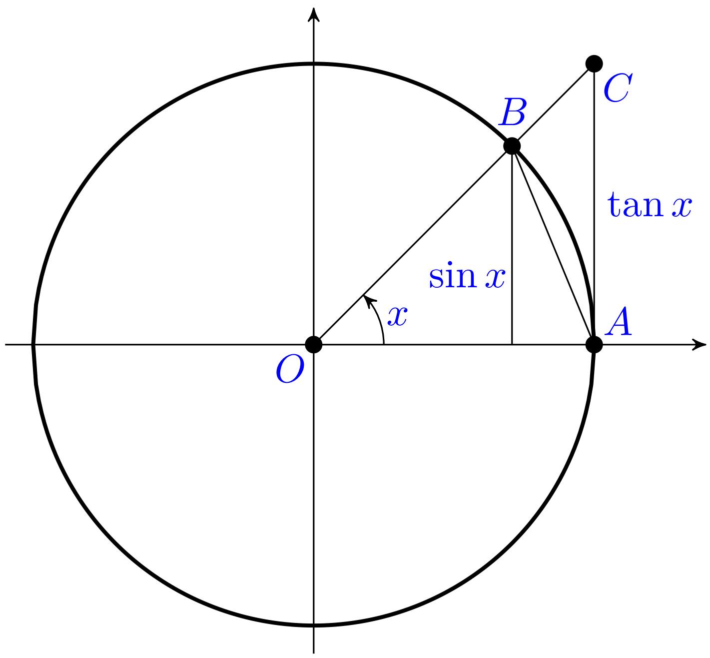


We first prove that if $0 < x < \frac{\pi}{2}$ then

$$\cos x < \frac{\sin x}{x} < 1.$$

Note that, referring to the picture on the right where the radius of the circle is R,

$$\text{area of triangle } AOB < \text{ area of sector } AOB < \text{ area of triangle } AOC,$$



and therefore

$$\frac{1}{2}R^2 \sin x < \frac{1}{2}R^2 x < \frac{1}{2}R^2 \tan x \Rightarrow \cos x < \frac{\sin x}{x} < 1.$$

Since replacing x by $-x$ the inequalities remain true, they hold for all x in $0 < |x| < \frac{\pi}{2}$. It also follows that $|\sin x| \leqslant |x|$ for all $x \in \mathbb{R}$: for $x = 0$ and $|x| \geqslant \frac{\pi}{2}$ this is obvious and for $0 < |x| < \frac{\pi}{2}$ it follows from $\frac{\sin x}{x} < 1$.

We have now that

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \leqslant 2\left|\sin\left(\frac{x}{2}\right)\right| \leqslant |x|.$$

Thus $\left|1 - \frac{\sin x}{x}\right| < |x|$ as long as $0 < |x| < \frac{\pi}{2}$. Given $\varepsilon > 0$, take $\delta = \min\{\varepsilon, \frac{\pi}{2}\}$ so $0 < |x| < \delta$ implies $\left|1 - \frac{\sin x}{x}\right| < \varepsilon$.

Example 7. For any $a \in \mathbb{R}$, $\lim\limits_{x \to a} \sin x = \sin a$.

Solution: Recall the formulas $\sin(u+v) = \sin u \cos v + \cos u \sin v$ and $\sin(u-v) = \sin u \cos v - \cos u \sin v$. Subtracting them we get $2\cos u \sin v = \sin(u+v) - \sin(u-v)$. Now let $\alpha = u + v$ and $\beta = u - v$ so $u = \frac{\alpha+\beta}{2}$ and $v = \frac{\alpha-\beta}{2}$. We thus obtain the formula $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$.

Therefore

$$|\sin x - \sin a| \leqslant 2\left|\sin\frac{x-a}{2}\right|\left|\cos\frac{x+a}{2}\right| \leqslant |x - a|.$$

(We have used the inequality $|\sin x| \leqslant |x|$ proven above.) One may thus take $\delta = \varepsilon$ in the definition of limit.

2.2 Nonexistence of a Limit

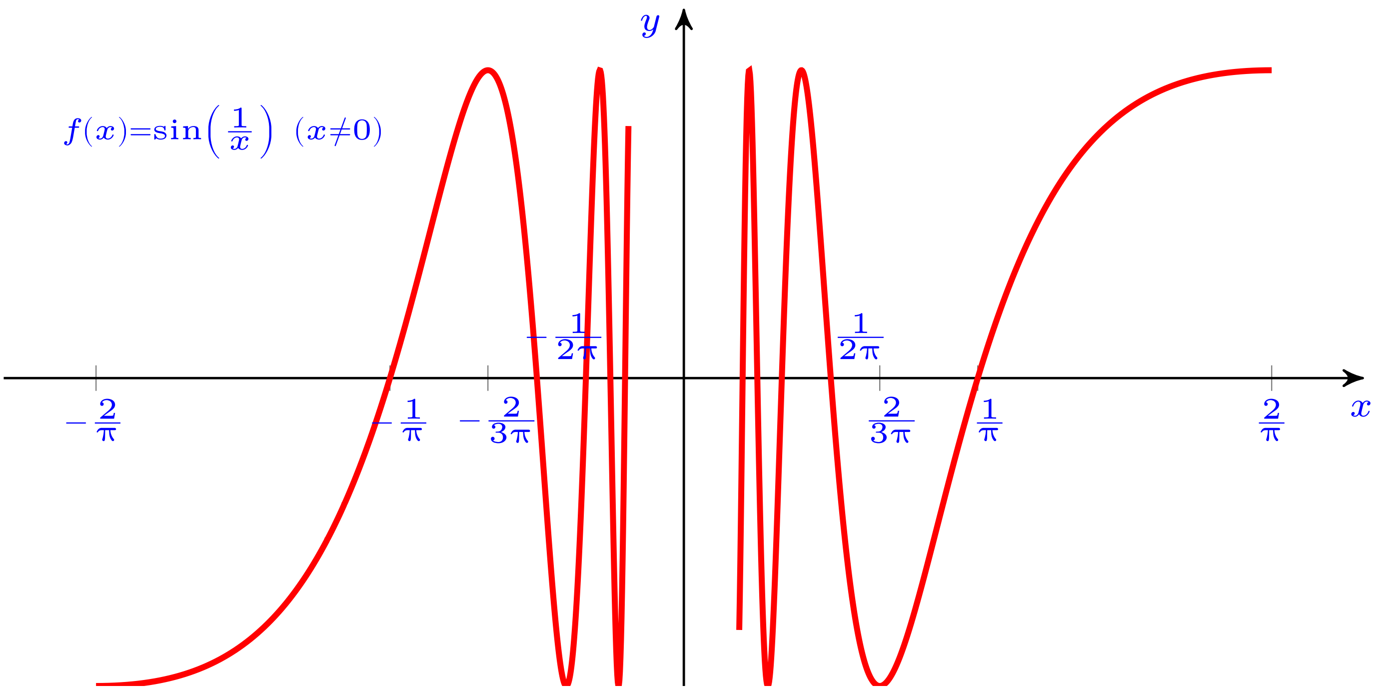
If $L \in \mathbb{R}$ is the limit of $f(x)$ at $a \in \mathbb{R}$ then for any fixed $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x', x'' \in (a-\delta, a+\delta) \setminus \{a\}$ we have $|f(x') - f(x'')| < \varepsilon$ (*stop and think why!*). Therefore if every deleted neighborhood of a contains a pair of points $x', x'' \in \mathbb{R}$ for which $|f(x') - f(x'')| \geqslant \varepsilon$ at least for *some* $\varepsilon > 0$ then f cannot have a limit at a.

Example 8. What is the limit of $f(x) = \frac{|x|}{x}$ at 0?

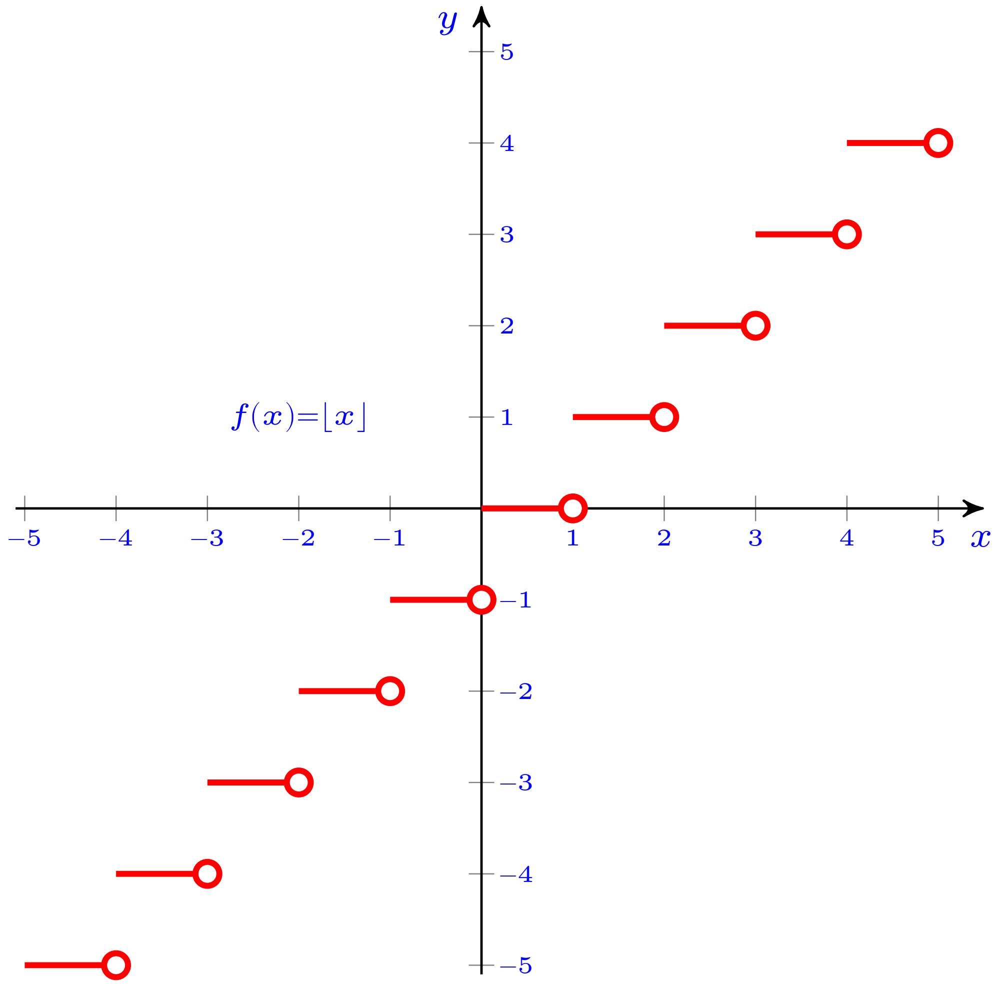
Example 9 (Dirichlet Function)**.** What is the limit of $D(x)$ at a point $a \in \mathbb{R}$ if

$$D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Example 10. What is the limit of $f(x) = \sin\left(\frac{1}{x}\right)$ at 0?



Example 11. This example introduces two functions that commonly arise in Calculus: for any $x \in \mathbb{R}$, $\lfloor x \rfloor$, called the *integer part* of x, is defined as the largest integer less than or equal to x. Also, by $\{x\}$ (here the curly braces $\{\}$ should not be confused with those denoting sets) we mean $x - \lfloor x \rfloor$. At what points do the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \{x\}$ have limits and what are their limits at these points?


y
x
f(x)=⌊x⌋
−5
−4
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1
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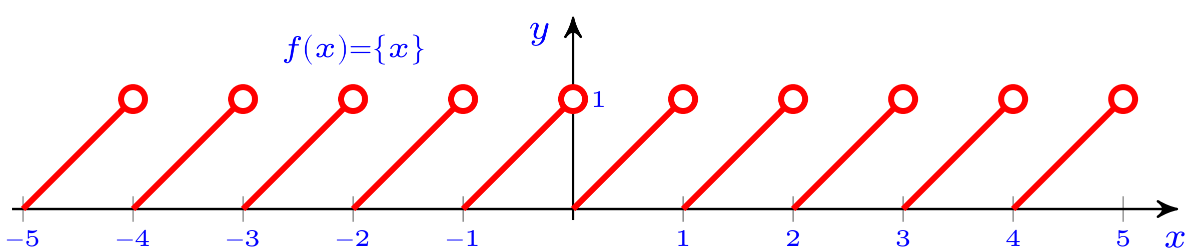

y
x
f(x)={x}
1
−5
−4
−3
−2
−1
1
2
3
4
5

2.3 Problems

1. Find the limit of the function
$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0 \end{cases}$$
at the point a) $x = -1$; b) $x = -0.001$; c) $x = 0$; d) $x = 0.01$.

2. Does the function
$$f(x) = \begin{cases} 3x & \text{if } -1 < x \leqslant 1, \\ 2x & \text{if } 1 < x \leqslant 3 \end{cases}$$
have a limit at a) $x = \frac{1}{2}$; b) $x = 1$; c) $x = 1.1$?

3. Prove that if $f(x)$ has a constant value k in some deleted neighborhood of a, then $\lim\limits_{x\to a} f(x) = k$.

4. Prove that changing the value of a function $f(x)$ at any point $x_1 \neq a$ has no effect on the limit of $f(x)$ at a (if any).

5. Use the definition of limit (the "ε-δ language") to prove that (a) $\lim\limits_{x\to 2} x^3 = 8$; (b) $\lim\limits_{x\to 2} \dfrac{5x^2 - 10x}{x-2} = 10$; (c) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2} = 1$.

Solution: For (c), we have
$$\left|1 - \frac{\sin^2 x}{x^2}\right| = \left|1 - \frac{\sin x}{x} + \frac{\sin x}{x} - \frac{\sin^2 x}{x^2}\right| \leqslant \left|1 - \frac{\sin x}{x}\right| + \left|\frac{\sin x}{x}\right| \cdot \left|1 - \frac{\sin x}{x}\right| < \varepsilon$$
as long as δ is chosen so that $0 < |x| < \delta$ implies $\left|1 - \dfrac{\sin x}{x}\right| < \dfrac{\varepsilon}{2}$, which can de done since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

6. Use the definition of limit (the "ε-δ language") to prove that for any $a \in \mathbb{R}$, (a) $\lim\limits_{x\to a} \cos x = \cos a$; (b) $\lim\limits_{x\to a} x^n = a^n$, $n \in \mathbb{N}$; (c) $\lim\limits_{x\to a} \tan x = \tan a$, $a, x \neq \frac{\pi}{2} + \pi n$, $n \in \mathbb{N}$.

Solution: For (b), we have
$$|x^n - a^n| \leqslant |x - a| \left(|x|^{n-1}|a| + |x|^{n-2}|a|^2 + \cdots + |x||a|^{n-1}\right).$$
If $|x - a| < |a|$ then $|x| < 2|a|$, and so
$$|x^n - a^n| < |x - a| \cdot \underbrace{|a|^n \cdot (2^{n-1} + 2^{n-2} + \cdots + 2)}_{M} < \varepsilon \Rightarrow |x - a| < \frac{\varepsilon}{M}.$$
We may now take $\delta = \min\{|a|, \frac{\varepsilon}{M}\}$.
For (c) we note first of all that
$$|\tan x - \tan a| = \left|\frac{\sin x}{\cos x} - \frac{\sin a}{\cos a}\right| = \frac{|\sin(x-a)|}{|\cos x||\cos a|} \leqslant \frac{|x-a|}{|\cos x||\cos a|}.$$
We now consider two cases: 1) If $a = \pi k$ for some $k \in \mathbb{Z}$ then we may take $\delta = \min\{\varepsilon|\cos(a-1)|, 1\}$ so $0 < |x - a| < \delta$ implies
$$|\tan x - \tan a| \leqslant \frac{|x-a|}{|\cos x|} < \frac{|\cos(a-1)|\varepsilon}{|\cos(a-1)|} = \varepsilon$$

since $|x-a|<1$ implies $|\cos x|>|\cos(a-1)|$. 2) If $a\neq \pi k$ for any $k\in\mathbb{Z}$, let $n\in\mathbb{Z}$ be such that $\frac{\pi}{2}+\pi n<a<\frac{\pi}{2}+\pi(n+1)$. Define

$$\mu=\frac{1}{2}\min\left\{\left|a-\left(\frac{\pi}{2}+\pi n\right)\right|,\left|a-\left(\frac{\pi}{2}+\pi(n+1)\right)\right|,|a-(\pi+\pi n)|\right\}$$

so that for any x with $|x-a|<\mu$ there holds

$$|\cos x|\geqslant M:=\min\{|\cos(a-\mu)|,|\cos(a+\mu)|\}>0.$$

Thus if $\delta=\min\{M\varepsilon|\cos a|,\mu\}$, then $0<|x-a|<\delta$ implies

$$|\tan x-\tan a|\leqslant\frac{|\sin(x-a)|}{|\cos x||\cos a|}\leqslant\frac{|x-a|}{|\cos x||\cos a|}<\frac{M\varepsilon|\cos a|}{|\cos a|}\cdot\frac{1}{M}=\varepsilon.$$

7. Formulate (preferably using logical symbolism) the statement that a function $f(x)$ does *not* approach the limit $L\in\mathbb{R}$ as $x\to a$.

8. Which of the following limits exist: (a) $\lim\limits_{x\to 0}\frac{x}{x}$; (b) $\lim\limits_{x\to 0}\frac{x|x|}{x}$; (c) $\lim\limits_{x\to 0}\frac{\sin\frac{1}{x}}{\sin\frac{1}{x}}$; (d) $\lim\limits_{x\to 0}\frac{\sin\frac{1}{|x|}}{\sin\frac{1}{x}}$?

9. Prove that a nonnegative function cannot have a negative limit at any point.

10. Let $D(x)$ be the Dirichlet function. Discuss the limiting behavior of (a) $xD(x)$; (b) $|x|D(x)$; (c) $\frac{|x|}{x}D(x)$; (d) $D(x)\sin x$.

Solution: (a) Since $|xD(x)|\leqslant|x|$, we have $\lim_{x\to 0}xD(x)=0$. For any $a\neq 0$, in any deleted neighborhood of a, let $x'\in\mathbb{Q}$ and $x''\in\mathbb{R}\setminus\mathbb{Q}$ so $|f(x')-f(x'')|=|x'|$, which cannot be made smaller than an arbitrary ε. Thus

$$\lim_{x\to a}xD(x)=\begin{cases}0, & \text{if } a=0,\\ \text{DNE}, & \text{if } a\neq 0.\end{cases}$$

For (b), the argument and result are identical.

For (c) we have

$$f(x)=\frac{|x|}{x}D(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}_{>0},\\ -1, & \text{if } x\in\mathbb{Q}_{<0},\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\Rightarrow\lim_{x\to a}f(x)=\text{DNE}\ \forall a\in\mathbb{R}.$$

For (d), the argument and result are identical to those of (a) and (b).

11. Show that $\lim_{x\to 0}x\cdot\lfloor\frac{1}{x}\rfloor=1$.

Solution: Given $\varepsilon>0$, let $n\in\mathbb{N}$ be the smallest with the property that $n>\frac{1}{\varepsilon}-1$ (if $\varepsilon>1$ then $n=1$.) By Archimedes' principle such an n exists. Note that this implies that $\frac{1}{n+1}<\varepsilon$. We claim that we can take $\delta = \frac{1}{n+1}$ in the definition of limit. Let $x \in \mathbb{R}$ be such that $0 < |x| < \delta$. If $x > 0$, let $m \in \mathbb{N}$ be such that $\frac{1}{m+1} < x \leqslant \frac{1}{m}$ so $m \geqslant n+1$. Then

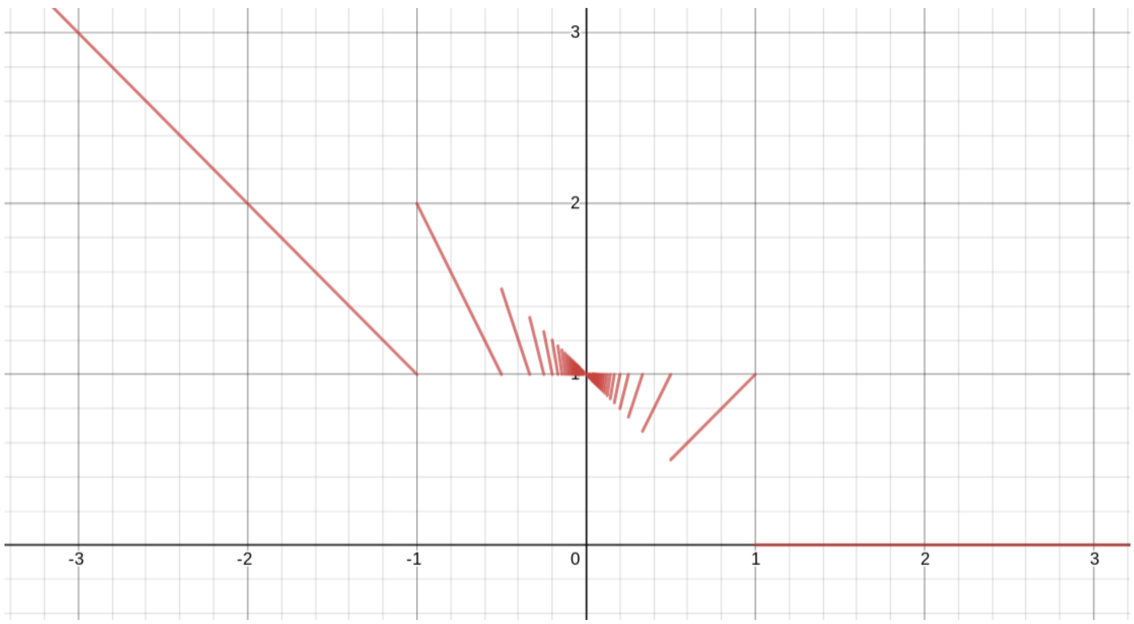
$$m \leqslant \frac{1}{x} < m+1 \Rightarrow \left\lfloor \frac{1}{x} \right\rfloor = m \Rightarrow \frac{m}{m+1} \leqslant x \left\lfloor \frac{1}{x} \right\rfloor \leqslant 1 \Rightarrow$$

$$\left| x \left\lfloor \frac{1}{x} \right\rfloor - 1 \right| \leqslant 1 - \frac{m}{m+1} = \frac{1}{m+1} < \frac{1}{m} \leqslant \frac{1}{n+1} < \varepsilon.$$

For $-\frac{1}{n+1} < x < 0$ the argument is similar. Let $m \in \mathbb{N}$ be such that $-\frac{1}{m} < x \leqslant -\frac{1}{m+1}$ so $m \geqslant n+1$. Then

$$-(m+1) \leqslant \frac{1}{x} < -m \Rightarrow \left\lfloor \frac{1}{x} \right\rfloor = -(m+1) \Rightarrow 1 \leqslant x \left\lfloor \frac{1}{x} \right\rfloor < 1 + \frac{1}{m} \Rightarrow \left| x \left\lfloor \frac{1}{x} \right\rfloor - 1 \right| < \frac{1}{m} \leqslant \frac{1}{n+1} = \varepsilon.$$

The graphs of $\left\lfloor \frac{1}{x} \right\rfloor$ and $x \left\lfloor \frac{1}{x} \right\rfloor$ (below) are worth thinking about.



3 Lecture 2 – Basic Properties of Limits

3.1 Theorems

Here as before we consider a real-valued function f defined on a subset D of $\mathbb{R}$.

Theorem 5 (Uniqueness of limit)**.** If $f \colon D \to \mathbb{R}$ has a limit $L \in \mathbb{R}$ at a limit point $a \in \mathbb{R}$ of D then the limit is unique.

Proof. Suppose that L and $L_1 \in \mathbb{R}$ are two numbers that both satisfy the definition of limit and that $L_1 \neq L_2$. Then for $\varepsilon := \frac{1}{2}|L_1 - L_2| > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for any $x \in D$ we have implications

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$$
$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. For any $x \in D$ satisfying $0 < |x - a| < \delta$ we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leqslant |L_1 - f(x)| + |f(x) - L_2| < 2\varepsilon = |L_1 - L_2|.$$

The obtained contradiction shows that the assumption $|L_1 - L_2| > 0$ is infeasible. ∎

Theorem 6 ("Preservation of sign")**.** If $f\colon D \to \mathbb{R}$ has a limit $0 \neq L \in \mathbb{R}$ at a limit point $a \in \mathbb{R}$ of D then there is a deleted neighborhood of a in which f has the same sign as L, that is, f is positive in the deleted δ-neighborhood of a if $L > 0$, and f is negative in the deleted δ-neighborhood of a if $L < 0$.

Proof. Suppose for certainty that $L > 0$ and let $\varepsilon = L/2$. Then there exists a $\delta > 0$ such that for all $x \in D$, $0 < |x - a| < \delta$ implies $0 < L/2 < f(x) < 3L/2$. In the case when $L < 0$ the proof is analogous. ∎

Theorem 7. The function $f\colon D \to \mathbb{R}$ has a limit $L \in \mathbb{R}$ at a limit point $a \in \mathbb{R}$ of D if and only if $f(x) = L + \alpha(x)$ where $\alpha\colon D \to \mathbb{R}$ has limit 0 at a.

Proof. Suppose that f has limit L at a. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\alpha(x)| = |f(x) - L| < \varepsilon$ and so $\alpha(x) \to 0$ as $x \to a$. If $f(x) = L + \alpha(x)$ with $\alpha(x) \to 0$ when $x \to a$, reversing the argument we conclude that $f(x) \to L$ as $x \to a$. ∎

NOTE 8. It is sometimes easier to show that $\lim\limits_{x\to a}(f(x) - L) = 0$ than to show that $\lim\limits_{x\to a} f(x) = L$. Theorem 7 says that these two things are logically equivalent..

DEFINITION 11 (Function bounded on a set)**.** A function $f\colon D \to \mathbb{R}$ is called *bounded* on a set $A \subseteq D$ if there exists a constant $M \in \mathbb{R}$ such that $|f(x)| \leqslant M$ for *all* $x \in A$.

Theorem 8 (Existence of limit implies boundness at limit point)**.** If $f\colon D \to \mathbb{R}$ has a finite limit at a limit point $a \in \mathbb{R}$ of D, then f is bounded in some deleted neighborhood of D.

Proof. Suppose that $\lim_{x\to a} f(x) = L \in \mathbb{R}$. Then for $\varepsilon = 1$ there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x)| - |L| \leqslant |f(x) - L| < 1$, and so $f(x)$ is bounded by $|L| + 1$ in this deleted δ-neighborhood of a. ∎

Theorem 9 ("Bounded times small equals small")**.** If $f\colon D \to \mathbb{R}$ has limit 0 at a limit point a of D, and a function g is bounded in some deleted neighborhood of a, then $\lim\limits_{x\to a} f(x)g(x) = 0$.

Proof. Given $\varepsilon > 0$, let $M \in \mathbb{R}$ and $\delta_1 > 0$ be such that $0 < |x - a| < \delta_1$ implies $|g(x)| < M$. Now let $\delta_2 > 0$ be such that $0 < |x-a| < \delta_2$ implies $|f(x)| < \frac{\varepsilon}{M}$. Then for $\delta = \min\{\delta_1, \delta_2\}$, $0 < |x-a| < \delta$ implies $|f(x)g(x)| < \frac{\varepsilon}{M}\cdot M = \varepsilon$. ∎

Theorem 10 (Arithmetic with Limits)**.** Suppose functions f and g are both defined in some deleted neighborhood of a point a.

1. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = S$ then $\lim_{x\to a} f(x) \pm g(x) = L \pm S$, that is,

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$

   More generally, if f_i, $1 \leqslant i \leqslant n$, are functions with a being a limit point of their domains, and $\lim_{x\to a} f_i(x) = L_i$, then

$$\lim_{x\to a}\sum_i f_i(x) = \sum_i \lim_{x\to a} f_i(x) = \sum_i L_i.$$

2. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = S$ then $\lim_{x\to a} f(x)g(x) = LS$, that is,

$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right).$$

   More generally, if f_i, $1 \leqslant i \leqslant n$, are functions with a being a limit point of their domains, and $\lim_{x\to a} f_i(x) = L_i$, then

$$\lim_{x\to a}\prod_i f_i(x) = \prod_i \lim_{x\to a} f_i(x) = \prod_i L_i.$$

3. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = S \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{S}$, that is,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$

Proof. For part 1, for a fixed $\varepsilon > 0$, let $\delta_1 > 0$ and $\delta_2 > 0$ be such that, respectively, $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ imply $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - S| < \frac{\varepsilon}{2}$. Then for any x satisfying $0 < |x-a| < \delta = \min\{\delta_1, \delta_2\}$,

$$|(f(x) \pm g(x)) - (L \pm S)| \leqslant |f(x) - L| + |g(x) - S| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

proving the desired result. For the more general result, apply the proven result repeatedly.

For part 2, applying Theorem 7 we have $f(x) = L + \alpha(x)$ and $g(x) = S + \beta(x)$ for some functions α and β with the property that $\lim_{x\to a} \alpha(x) = \lim_{x\to a} \beta(x) = 0$. Now

$$fg = LS + (L\beta + S\alpha + \alpha\beta),$$

and all three terms in parentheses approach zero (*why?*). Therefore, by part 1, $\lim_{x\to a} fg = LS$.

For part 3, we have (applying part 2) $\lim(S \cdot g(x)) = S^2$, and so $Sg(x)$ is bounded in some deleted neighborhood of a. More precisely, for $\varepsilon = \frac{1}{2}S^2$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $\frac{1}{2}S^2 < Sg(x) < \frac{3}{2}S^2$. Therefore $\frac{2}{3S^2} < \frac{1}{Sg(x)} < \frac{2}{S^2}$. Applying again part 2, we conclude that $\lim_{x\to a}(fS - gL) = 0$. Therefore, by Theorem 9

$$\lim_{x\to a} \left(\frac{f}{g} - \frac{L}{S}\right) = \frac{fS - gL}{gS} = 0,$$

and so by Theorem 7, $\frac{f}{g} \to \frac{L}{S}$ as $x \to a$. ■

Theorem 11 (Limit of a Composite Function)**.** Suppose a is a limit point of the domain of the function g, $g(x) \to \ell$ as $x \to a$ and $g(x) \neq \ell$ in some deleted neighborhood of a. Suppose further that ℓ is a limit point of the domain of f and $f(x) \to L$ as $x \to \ell$. Then $f \circ g$ has limit L as $x \to a$.

Proof. Given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$0 < |g(x) - \ell| < \delta_1 \Rightarrow |f(g(x)) - L| < \varepsilon.$$

For this δ_1 there exists $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - \ell| < \delta_1.$$

Also there exists some $\delta_3 > 0$ such that

$$0 < |x-a| < \delta_3 \Rightarrow |g(x) - \ell| > 0.$$

Now for $\delta = \min\{\delta_2, \delta_3\}$ we have

$$0 < |x-a| < \delta \Rightarrow 0 < |g(x) - \ell| < \delta_1,$$

and therefore

$$0 < |x-a| < \delta \Rightarrow |f(g(x)) - L| < \varepsilon.$$

Since ε is arbitrary it follows that $\lim_{x\to a} f(g(x)) = L$. ■

NOTE 9. Theorem 11 essentially makes it possible to *change variables* while computing limits. That is, with the notation and under the assumptions of the theorem,

$$\lim_{x\to a} f(g(x)) = \lim_{t\to \ell} f(t),$$

where $t := g(x)$. This gives a powerful practical tool for computing limits.

NOTE 10. In Theorem 11 one cannot dispense from the requirement that $g(x) \neq \ell$ in some neighborhood of a. The following example explains why. Consider

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x, & \text{if } x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right), k \in \mathbb{Z}, k \neq 0, \\ -x, & \text{if } x \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right), k \in \mathbb{Z}, k \neq 0, \\ 0, & \text{if } x = \frac{1}{k}, k \in \mathbb{Z}, k \neq 0 \end{cases}$$

We have $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$ since $|g(x)| \leqslant |x|$. However, in any deleted neighborhood of 0 we can take x' such that $g(x') \neq 0$ and $x'' = \frac{1}{k}$, $k \in \mathbb{Z} \setminus \{0\}$, then $|f(g(x')) - f(g(x''))| = 1$ cannot be made infinitesimally small. Therefore $\lim_{x\to 0} f(g(x))$ does not exist.

Take a look at examples 12 and 13.

Theorem 12 ("Squezze/sandwich")**.** Suppose functions f, g, and h are defined in some deleted neighborhood of a point a. Suppose also that in this deleted neighborhood we have $g(x) \leqslant f(x) \leqslant h(x)$ and also that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} f(x) = L$.

Proof. Fix an $\varepsilon > 0$. There exist $\delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$ for which we have the implications

$$\left.\begin{array}{l} 0 < |x-a| < \delta_1 \Rightarrow L - \varepsilon < g(x) < L + \varepsilon \\ 0 < |x-a| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon \\ 0 < |x-a| < \delta_3 \Rightarrow g(x) \leqslant f(x) \leqslant h(x) \end{array}\right\} \Rightarrow (0 < |x-a| < \min\{\delta_1, \delta_2, \delta_3\} L - \varepsilon < g(x) \leqslant f(x) \leqslant h(x) < L + \varepsilon),$$

and so $\lim_{x\to a} f(x) = L$, as claimed. ■

Theorem 13 (Comparison)**.** Suppose that functions f and g are defined in some deleted neighborhood of a point a. Suppose further that in that deleted neighborhood we have $f(x) \leqslant g(x)$, and that the limits of both f and g at a exist. Then $\lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x)$.

Proof. We could use a fact from the previous problem set: if a function $f(x) \geqslant 0$ for all x in a deleted neighborhood of a point a, which is a limit point of the domain of f, and $L := \lim_{x\to a} f(x)$ exists, then $L \geqslant 0$. But we prefer a proof from scratch.

Let $L = \lim_{x\to a} f(x)$ and $S = \lim_{x\to a} g(x)$. We are to prove that $L \leqslant S$. Suppose by contradiction that $L > S$, and let ε be such that $S - L + 2\varepsilon < 0$. Let $\delta_1, \delta_2, \delta_3 > 0$ be such that

$$\left.\begin{array}{l} 0 < |x-a| < \delta_1 \Rightarrow g(x) - f(x) \geqslant 0 \\ 0 < |x-a| < \delta_2 \Rightarrow S - \varepsilon < g(x) < S + \varepsilon \\ 0 < |x-a| < \delta_3 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon \end{array}\right\} \Rightarrow (0 < |x-a| < \min\{\delta_1, \delta_2, \delta_3\} \Rightarrow g(x) - f(x) < S - L + 2\varepsilon < 0),$$

contradicting $g(x) \geqslant f(x)$. ■

NOTE 11. If in Theorem 13 we replace the non-strict inequality $f(x) \leqslant g(x)$ by the strict one $f(x) < g(x)$ then we would still have $\lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x)$. However it is *not* true that $f(x) < g(x) \Rightarrow \lim_{x\to a} f(x) < \lim_{x\to a} g(x)$. For example, if $f(x) = 0$ and $g(x) = x^2$, then $\lim_{x\to 0} f(x) = 0 < 0 = \lim_{x\to a} g(x)$ is false.

3.2 Examples

Example 12 (Polynomial and rational functions)**.** For any $a \in \mathbb{R}$, $n \in \mathbb{N}$, and $a_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$, $a_n \neq 0$, let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then

$$\lim_{x\to a} f(x) = f(a).$$

This follows immediately from Theorem 10, parts 1 and 2. Also let $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, $m \in \mathbb{N}$, $b_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$, $b_m \neq 0$. If $g(a) \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$. Here we have used Theorem 3. Here is one numerical example:

$$\lim_{x\to -1} \frac{x^3 - 1}{x^2 - 4x + 3} = \frac{(-1)^3 - 1}{(-1)^2 - 4\cdot(-1) + 3} = -\frac{1}{4}.$$

Example 13. Compute the limits (here $b \neq 0$): (a) $\lim_{x \to 0} \frac{\sin ax}{bx}$; (b) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$; (c) $\lim_{x \to 0} \frac{\sin(\sin x)}{x}$; (d) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

Example 14. Here we establish an important result: for any $r \in \mathbb{Q}$[2],

$$\lim_{x\to 0} \frac{(1+x)^r - 1}{x} = r.$$

If $r = n \in \mathbb{N}$ then by the Binomial Theorem,

$$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to 0} \frac{1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + x^n - 1}{x} = n.$$

If $r = \frac{1}{m}$, $m \in \mathbb{N}$, we first verify that

$$1 - |x| \leqslant \sqrt[m]{1+x} \leqslant 1 + |x|, \quad |x| < 1.$$

For the first inequality: if $x \geqslant 0$ then we are done. If $-1 < x < 0$, the $1 - |x| = 1 + x > 0$, and the inequality is equivalent to $(1+x)^{m-1} \leqslant 1$, which is true. For the second inequality: if $x < 0$ then we are done, and if $0 \leqslant x < 1$ then the inequality is equivalent to $(1+x)^{m-1} \geqslant 1$, which is true.

Now taking $y = (1+x)^{\frac{1}{m}} - 1$, so $x = (1+y)^m - 1$, we see that $\lim_{x\to 0} y = \lim_{x\to 0}(\sqrt[m]{1+x} - 1) = 0$. Then

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{m}} - 1}{x} = \lim_{y\to 0} \frac{y}{(1+y)^m - 1} = \frac{1}{\lim_{y\to 0} \frac{(1+y)^m - 1}{y}} = \frac{1}{m}.$$

Finally, in the general case $r = \frac{n}{m}$, taking $y = \sqrt[m]{1+x} - 1$, we have $x = (1+y)^m - 1$ and $(1+x)^{\frac{n}{m}} = (1+y)^n$, so

$$\lim_{x\to 0} \frac{(1+x)^{\frac{n}{m}} - 1}{x} = \lim_{y\to 0} \frac{(1+y)^n - 1}{(1+y)^m - 1} = \frac{n}{m}.$$

Note: Since $\lim_{x\to 0} \sqrt[n]{1+x} = 1$ for any $n \in \mathbb{N}$, it follows that $\lim_{x\to 1} \sqrt[n]{x} = 1$.

Example 15. Evaluate $\lim_{x\to 0} \sin x \sin \frac{1}{x}$.

Example 16. Evaluate $\lim\limits_{x\to 5} \dfrac{2x^2 - 11x + 5}{3x^2 - 14x - 5}$.

[2] Later we we will show that the result holds for any $r \in \mathbb{R}$.

Example 17. Evaluate $\lim\limits_{x\to\pi} \dfrac{\sin^2 x}{1+\cos^3 x}$.

Example 18. Evaluate $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-\sqrt[3]{x}}$.

There are two ideas to explore here: 1) take $\sqrt[6]{x} = t$, use the Limit of Composite Function Theorem and the fact that $\lim\limits_{x\to 1} \sqrt[n]{x} = 1$; 2) multiply top and bottom by $(1+\sqrt{x})(1+\sqrt[3]{x}+\sqrt[3]{x^2})$.

Example 19. Evaluate $\lim\limits_{x\to 1} \dfrac{\cos\frac{\pi x}{2}}{1-x}$.

3.3 Problems

1. Prove that if $\lim_{x\to x_0} f(x) = c$, then $\lim_{x\to x_0} |f(x)| = |c|$. Is the converse true?

2. One has to be very careful when using the arithmetic of limits. See again Note 10. Here are some other warnings.

   - Show that the existence of the limit $\lim_{x\to a}[f(x) + g(x)]$ does not imply the existence of the limits $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$.
   - Show that the existence of the limit $\lim_{x\to a} f(x)g(x)$ does not imply the existence of the limits $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$.
   - Note 10 shows that the existence of limits of f and g does not imply the existence of the limit of $f(g(x))$. The converse is also false! That is, the existence of the limit $\lim_{x\to a} f(g(x))$ does not imply the existence of the limits $\lim_{x\to a} g(x)$ or $\lim_{x\to \ell} f(x)$ for any $\ell \in \mathbb{R}$. Consider $D(D(x))$ where $D(x)$, is the Dirichlet function.

3. Show that
$$\lim_{x\to 0} \sin(\cos x)\sin x = 0.$$
How about
$$\lim_{x\to 0} \sin(\cot^2 x)\sin x?$$

4. Prove that if
$$\lim_{x\to a} f(x) < \lim_{x\to a} g(x),$$
then there is a deleted neighborhood of a in which $f(x) < g(x)$.

5. Evaluate

   (a) $\displaystyle\lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x)-1}{x}$;

   (b) $\displaystyle\lim_{x\to 0} \frac{(1+x)^5-(1+5x)}{x^2+x^5}$;

   (c) $\displaystyle\lim_{x\to 3} \frac{x^2-5x+6}{x^2-8x+15}$;

   (d) $\displaystyle\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$.

6. Evaluate

   (a) $\displaystyle\lim_{x\to 1} \frac{x^4-3x+2}{x^5-4x+3}$;

   (b) $\displaystyle\lim_{x\to 2} \frac{x^3-2x^2-4x+8}{x^4-8x^2+16}$;

   (c) $\displaystyle\lim_{x\to -1} \frac{x^3-2x-1}{x^5-2x-1}$;

   (d) $\displaystyle\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$.

7. Let m and n be arbitrary positive integers. Evaluate

   (a) $\displaystyle\lim_{x\to 1} \frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$;

Solution: Performing polynomial long division twice we find that

$$x^{n+1} - (n+1)x + n = (x-1)(x^n + x^{n-1} + \cdots + x - n) = (x-1)^2(x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n).$$

Therefore,

$$\lim_{x\to 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

(b) $\lim_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$.

8. Evaluate the following limits, where m and n are arbitrary integers:

(a) $\lim_{x\to a} \dfrac{(x^n - a^n) - na^{n-1}(x-a)}{(x-a)^2}$;

Solution: We have for any $x \neq a$:

$$\begin{aligned}\frac{(x^n - a^n) - na^{n-1}(x-a)}{(x-a)^2} &= \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) - na^{n-1}(x-a)}{(x-a)^2} \\ &= \frac{x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + (1-n)a^{n-1}}{(x-a)} \\ &= x^{n-2} + 2x^{n-3}a + \cdots + (n-2)xa^{n-3} + (n-1)a^{n-2}.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{x\to a} \frac{(x^n - a^n) - na^{n-1}(x-a)}{(x-a)^2} &= \left(x^{n-2} + 2x^{n-3}a + \cdots + (n-2)xa^{n-3} + (n-1)a^{n-2}\right)\Big|_{x=a} \\ &= a^{n-2}(1 + 2 + \cdots + (n-1)) \\ &= \frac{a^{n-2}n(n-1)}{2}.\end{aligned}$$

(b) $\lim_{x\to 0} \dfrac{(1+mx)^n - (1+nx)^m}{x^2}$.

Solution: We have

$$\begin{aligned}\frac{(1+mx)^n - (1+nx)^m}{x^2} &= \frac{1 + nmx + \frac{n(n-1)}{2}(mx)^2 + \binom{n}{3}(mx)^3 + \cdots + (mx)^n}{x^2} \\ &\quad - \frac{1 + nmx + \frac{m(m-1)}{2}(nx)^2 + \binom{m}{3}(nx)^3 + \cdots + (nx)^m}{x^2} \\ &= \frac{n(n-1)}{2} - \frac{m(m-1)}{2} + \left(\binom{n}{3} - \binom{m}{3}\right)x + \cdots.\end{aligned}$$

Therefore,

$$\lim_{x\to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2} = \frac{n(n-1)}{2} - \frac{m(m-1)}{2}.$$

(c) $\lim_{x\to 1} \left(\dfrac{m}{1-x^m} - \dfrac{n}{1-x^n}\right)$.

Solution: First of all note that if $m = n$, the limit of this expression is 0. Assume now that $m > n$. Note that

$$\begin{aligned}\frac{m}{1-x^m} - \frac{n}{1-x^n} &= \frac{m(1-x^n) - n(1-x^m)}{(1-x^m)(1-x^n)} \\ &= \frac{mm(1-x)(1+x+x^2+\cdots+x^{n-1}) - n(1-x)(1+x+x^2+\cdots+x^{m-1})}{(1-x)(1-x)(1+x+x^2+\cdots+x^{m-1})(1+x+x^2+\cdots+x^{n-1})} \\ &= \frac{m(1+x+x^2+\cdots+x^{n-1}) - n(1+x+x^2+\cdots+x^{m-1})}{(1-x)(1+x+x^2+\cdots+x^{m-1})(1+x+x^2+\cdots+x^{n-1})}.\end{aligned}$$

Note that the numerator and denominator of this fraction still turn 0 when $x = 1$. Under the assumption $m > n$ the polynomial in the numerator is of degree $m-1$, and it is a (somewhat tedious) computation involving long division to check that

$$\begin{aligned}&-nx^{m-1} - nx^{m-2} - \cdots - nx^{n+1} - nx^n + (m-n)x^{n-1} + (m-n)x^{n-2} + \cdots + (m-n)x + (m-n) \\ &= (x-1)(-nx^{m-2} - 2nx^{m-3} - \cdots - (m-n)nx^{n-1} - (m-n)(n-1)x^{n-2} - \cdots - (m-n)2x - (m-n))\end{aligned}$$

Taking the second factor as $f(x)$, we compute that $f(1) = \frac{mn(n-m)}{2}$. Therefore,

$$\lim_{x\to 1}\left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right) = -\frac{\frac{mn(n-m)}{2}}{mn} = \frac{m-n}{2}.$$

For the case $n > m$ we use symmetry to argue that the answer is $-\frac{m-n}{2} = \frac{n-m}{2}$.

9. Evaluate
   (a) $\displaystyle\lim_{x\to\pi} \frac{\sin mx}{\sin nx}$, $n \neq 0$;
   (b) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$;
   (c) $\displaystyle\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x}$;
   (d) $\displaystyle\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x)$.

10. Evaluate
    (a) $\displaystyle\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2}$;
    (b) $\displaystyle\lim_{x\to\frac{\pi}{2}} \tan 2x \tan\left(\frac{\pi}{4} - x\right)$;
    (c) $\displaystyle\lim_{x\to\frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\cos x}$;
    (d) $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{1 - \tan^2 x}$.

11. Evaluate
    (a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{1 - \sqrt{1 + \tan x}}$;
    (b) $\displaystyle\lim_{x\to 4} \frac{2 - \sqrt{x}}{3 - \sqrt{2x+1}}$;

(c) $\lim_{x\to -8} \frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$;

(d) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}-\sqrt{x-a}}{\sqrt{x^2-a^2}}$, $(a>0)$.

12. Evaluate

(a) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$;

(b) $\lim_{x\to 0} \frac{\sqrt{1-2x-x^2}-(1+x)}{x}$;

(c) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$.

4 Lecture 3 – One-Sided Limits

4.1 Definitions

Here as before we consider a real-valued function f defined on a subset D of $\mathbb{R}$.

DEFINITION 12 (Right-hand Limit)**.** Suppose a is a limit point of D. We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches a *from the right*",

or

"f has *right-hand limit* L as x tends to a",

and write

$$f(x) \to L \quad \text{as} \quad x \to a+,$$

or

$$f(x) \xrightarrow[x\to a+]{} L,$$

or

$$\lim_{x\to a+} f(x) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < x - a < \delta$, we have $|f(x) - L| < \varepsilon$.

In logical symbolism this definition is written as

$$\lim_{D\ni x\to a+} f(x) = L :=\forall\varepsilon > 0\ \exists\delta > 0\ \forall x \in D\ \ (0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon).$$

The definition of left-hand limit is similar.

DEFINITION 13 (Left-hand Limit)**.** Suppose a is a limit point of D. We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches a *from the left*",

or

"f has *left-hand limit* L as x tends to a", and write

$$f(x) \to L \quad \text{as} \quad x \to a-\,,$$

or

$$f(x) \xrightarrow[x\to a-]{} L,$$

or

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $-\delta < x - a < 0$ we have $|f(x) - L| < \varepsilon$.

In logical symbolism this definition is written as

$$\lim_{D\ni x\to a-} f(x) = L :\Leftrightarrow \forall\varepsilon > 0\ \exists\delta > 0\ \forall x \in D\ \left(-\delta < x - a < 0 \Rightarrow |f(x) - L| < \varepsilon\right).$$

NOTE 12. The differences between these two Definitions and the ε-δ definition of limit given in Lecture 1 is summarizes the following table.

Limit	Right-hand Limit	Left-hand Limit
a deleted neighborhood of the point a	an open interval $(a, a+h)$, $h > 0$	an open interval $(a-h, a)$, $h > 0$
x approaches a	x approaches a from the right	x approaches a from the left
$0 < \|x - a\| < \delta$	$0 < x - a < \delta$	$-\delta < x - a < 0$
$x \to a$	$x \to a+$	$x \to a-$

NOTE 13. Right-hand limits and left-hand limits are called *one-sided* limits. It should be clear that if $f(x)$ has a right-hand limit (left-hand limit) at a then this limit is unique. The proof is completely identical to the proof of the uniqueness of limit.

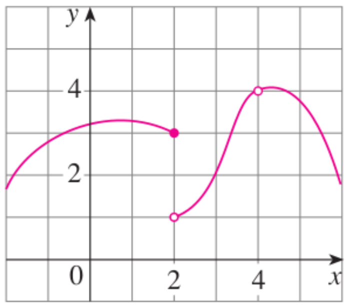
Theorem 14 (Existence of Limit)**.** Suppose a is a limit point of D. Then f has limit as $x \to a$ if and only if both the right-hand side limit at a and the left-hand side limit at a exist and are equal.

Proof. Suppose f has limit L at $x \to a$. That is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x \in D$ satisfying $0 < |x - a| < \delta$ we have $|f(x) - L| < \varepsilon$. Obviously this means that both $0 < x - a < \delta$ and $-\delta < x - a < 0$ imply $|f(x) - L| < \varepsilon$. Hence L is both the right-hand limit at a and the left-hand limit at a.

Reversing the argument we see that if $f(x)$ has L as the left-hand limit at a and as the right-hand limit at a, then $x \in D$, $0 < |x - a| < \delta$ imply $|f(x) - L| < \varepsilon$, and so $\lim f(x) = L$ as $x \to a$. (This requires selecting the minimum of two deltas.) ∎

4.2 Examples

Example 20. Using the figure provided, compute the following limits and quantities or state that they don't exist: (a) $\lim_{x\to 2-} f(x)$; (b) $\lim_{x\to 2+} f(x)$; (c) $\lim_{x\to 2} f(x)$; (d) $f(2)$; (e) $\lim_{x\to 4} f(x)$; (f) $f(4)$.



Example 21. Investigate the existence of limits and one-sided limits of $[x]$ and $\{x\}$ at a for all $a \in \mathbb{R}$.

4.3 Problems

1. Find the one-sided limits at $x = 0$ of the function
$$f(x) = \frac{x + x^2}{|x|}, \quad x \neq 0.$$

2. Find the one-sided limits at $x = 2$ of the function
$$f(x) = \begin{cases} x^2 & \text{if } -1 < x < 2, \\ 2x + 1 & \text{if } 2 \leqslant x < 3. \end{cases}$$

3. Find the one-sided limits of the function
$$f(x) = \begin{cases} -\dfrac{1}{1-x} & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ x & \text{if } 0 < x < 1, \\ 2 & \text{if } 1 \leqslant x. \end{cases}$$
at the point $x = 0$ and $x = 1$.

4. Does the function
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x < 0, \\ \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$
have a limit at $x = 0$? At $x = 1/\pi$? How about one-sided limits?

5. Does the function
$$f(x) = \left\lfloor \frac{1}{x} \right\rfloor$$
have a limit at $x = 0$? How about one-sided limits?

6. Prove that
$$\lim_{x \to 0} x \left\lfloor \frac{1}{x} \right\rfloor = 1.$$

7. Does the function
$$f(x) = \frac{\lfloor x \rfloor}{x}, \quad x \neq 0$$
have a limit at $x = 0$? How about one-sided limits?

8. State and prove analogues of all theorems concerning the arithmetic of limits for *one-sided* limits.

9. Prove that the sum of two functions, each with a one-sided limit at a, need not have a one-sided limit at a.

10. **Requires the completeness property of the real numbers.** Let f be bounded and increasing in an open nonempty interval (α, β). Prove that
$$\lim_{x \to a-} f(x) \text{ and } \lim_{x \to a+} f(x)$$
both exist for any $a \in (\alpha, \beta)$ and satisfy the inequality
$$\lim_{x \to a-} f(x) \leqslant \lim_{x \to a+} f(x).$$

Solution: Since the $\{f(x) : x \in (a, \beta)\}$ is not empty and is bounded from below (say, by $f(a)$), it has a (unique) infinnum. Call it R. We claim that R is the right-hand limit of f at a. From the definition of infinum, for any $\varepsilon > 0$, there exists $x' \in (a, \beta)$ such that $f(x') < R + \varepsilon$. Let $\delta = x' - a$. Then for any x with $a < x < a + \delta$ we have $f(x) < f(x') < R + \varepsilon$ and $f(x) \geqslant R > R - \varepsilon$. Thus

$$\lim_{x \to a+} f(x) = R.$$

The argument for the left-hand limit is quite similar. The set $\{f(x) : x \in (\alpha, a)\}$ is not empty and bounded from above (say, by $f(a)$). Let L be its supremum. Then for every $\varepsilon > 0$ there exists $x'' \in (\alpha, a)$ such that $f(x'') > L - \varepsilon$. Taking $\delta = a - x''$ we see that $a - \delta < x < a$ implies $L - \varepsilon < f(x) \leqslant L < L + \varepsilon$. Thus

$$\lim_{x \to a-} f(x) = L.$$

To show that $L \leqslant R$ assume by contradiction that $L > R$ and let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{2}(L - R)$. Then there exist $x_L \in (\alpha, a)$ and $x_R \in (a, \beta)$ such that

$$f(x_L) > L - \varepsilon > R + \varepsilon > f(x_R)$$

contradicting f being monotone increasing.

11. **Requires the concept of countable and uncountable sets.** Let f be bounded and increasing in a non-empty open interval (α, β). Prove that the number of points a where f makes a "jump", i.e. where

$$\lim_{x \to a-} f(x) < \lim_{x \to a+} f(x)$$

is countable. Construct an example of a function which is increasing, unbounded, and the set of points where it makes a jump is uncountable.

Solution: If $|f(x)| \leqslant M$ for all $x \in \mathrm{Dom}(f)$, argue that the set of points where the "height of the jump" is greater or equal than $M/2^k$, $k \in \mathbb{N}$, has cardinality at most $2^{k+1} - 1$, and so it is finite. The union of all such sets contains all points where f makes a jump.

12. State and prove the analogues of the previous two problems for (a) a decreasing function f; (b) a closed interval $[\alpha, \beta]$.

5 Lecture 4 – Infinite Limits

5.1 Definitions

Here as before we consider a real-valued function f defined on a subset D of $\mathbb{R}$. We also suppose that a is a limit point of D.

DEFINITION 14 (Infinite Limit)**.** We say that

"f approaches (or tends to) ∞ as x approaches a,

and write

$$f(x) \to \infty \quad \text{as} \quad x \to a,$$

or

$$f(x) \xrightarrow[x \to a]{} \infty,$$

or

$$\lim_{x\to a} f(x) = \infty$$

if for every $M > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < |x - a| < \delta$, we have $|f(x)| > M$.

In logical symbolism this definition is written as

$$\lim_{D\ni x\to a} f(x) = \infty := \forall M > 0\ \exists\delta > 0\ \forall x \in D\ \ (0 < |x - a| < \delta \Rightarrow |f(x)| > M)\,.$$

The definitions of limit equal to "$+\infty$" or "$-\infty$" are similar.

DEFINITION 15 (Infinite Limit "$+\infty$")**.** We say that

"f approaches (or tends to) $+\infty$ as x approaches a,

and write

$$f(x) \to +\infty \quad \text{as} \quad x \to a,$$

or

$$f(x) \xrightarrow[x\to a]{} +\infty,$$

or

$$\lim_{x\to a} f(x) = +\infty$$

if for every $M > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < |x - a| < \delta$, we have $f(x) > M$.

In logical symbolism this definition is written as

$$\lim_{D\ni x\to a} f(x) = +\infty := \forall M > 0\ \exists\delta > 0\ \forall x \in D\ \ (0 < |x - a| < \delta \Rightarrow f(x) > M)\,.$$

DEFINITION 16 (Infinite Limit "$-\infty$")**.** We say that

"f approaches (or tends to) $+\infty$ as x approaches a,

and write

$$f(x) \to -\infty \quad \text{as} \quad x \to a,$$

or

$$f(x) \xrightarrow[x\to a]{} -\infty,$$

or

$$\lim_{x\to a} f(x) = -\infty$$

if for every $M > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < |x - a| < \delta$, we have $f(x) < -M$.

In logical symbolism this definition is written as

$$\lim_{D\ni x\to a} f(x) = -\infty := \forall M > 0\ \exists\delta > 0\ \forall x \in D\ \ (0 < |x - a| < \delta \Rightarrow f(x) < -M)\,.$$

NOTE 14. The choice of the letters M and δ in the definitions above is a matter of custom, M being a symbol for a typical large number and δ a symbol for a typical small number (δ gets smaller as M gets larger.) Thus the built-in connotation of the phrase "given any $M > 0$" is "given any $M > 0$, however large."

Problem 1. Consider solving Problem 1 before reading further.

Go to examples 22 – 26.

5.2 Properties of Infinite Limits

Theorem 15. The function $f(x) \xrightarrow[x\to a]{} \infty$ if and only if $1/f(x) \xrightarrow[x\to a]{} 0$.

Proof. Suppose $\frac{1}{f} \xrightarrow[x\to a]{} 0$. Then given $M > 0$ there exists $\delta > 0$ such that for any $x \in \mathrm{Dom}(f)$, $0 < |x - a| < \delta \implies |1/f(x)| < \frac{1}{M}$, and so $|f(x)| > M$. Therefore $f(x) \xrightarrow[x\to a]{} \infty$. Reversing the argument we prove the converse. ∎

Theorem 16. Suppose f is bounded in some deleted neighborhood of a, and $g(x) \xrightarrow[x\to a]{} \infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0, \quad \lim_{x\to a} \frac{g(x)}{f(x)} = \infty.$$

COROLLARY 1 **($L \cdot \infty = \infty$ if $L \neq 0$).** If $f(x) \to L \neq 0$ as $x \to a$ (the case $L = \infty$ is allowed) and $g(x) \to \infty$ as $x \to a$, then $f(x)g(x) \to \infty$ as $x \to a$.

Proof. Since $1/f(x)$ has a finite limit as $x \to a$, it is bounded in some deleted neighborhood of a. By the second formula in the previous theorem $g(x)/(1/f(x)) = f(x)g(x) \to \infty$. ∎

COROLLARY 2 **($\frac{L}{0} = \infty$ if $L \neq 0$).** If $f(x) \to L \neq 0$ as $x \to a$ (the case $L = \infty$ is allowed) and $g(x) \to 0$ as $x \to a$, then $f(x)/g(x) \to \infty$ as $x \to a$.

COROLLARY 3 **($\frac{L}{\infty} = 0$ if $L \neq \infty$).** If $f(x) \to L \neq \infty$ and $g(x) \to \infty$ as $x \to a$ then $f(x)/g(x) \to 0$ as $x \to a$.

NOTE 15. Written *symbolically* these three corollaries say that

$$\begin{aligned} L \cdot \infty &= \infty, && \text{if } L \neq 0, \\ \frac{L}{0} &= \infty, && \text{if } L \neq 0, \\ \frac{L}{\infty} &= 0, && \text{if } L \neq \infty. \end{aligned}$$

The question however remains regarding the expressions

$$0 \cdot \infty, \ \frac{0}{0}, \ \frac{\infty}{\infty},$$

and some others, such as $\infty - \infty$, ∞^0, or 1^∞. These are called *intedeterminates.* See Example 27 for details.

Theorem 17 **($\infty + L = \infty$; $\infty + \infty = \infty$).** If $f(x) \to \infty$ and $g(x) \to L \in \mathbb{R}$ as $x \to a$ then $f(x) + g(x) \to \infty$ as $x \to a$. The same result holds if L is replaced by ∞ provided that $f(x)$ and $g(x)$ have the same sing in some deleted neighborhood of a.

Proof. We have

$$f(x) + g(x) = f(x)\left(1 + \frac{g(x)}{f(x)}\right),$$

where by Corollary 3 $g(x)/f(x) \to 0$ as $x \to a$. Therefore $1 + g(x)/f(x) \to 1$ as $x \to a$. Now by Corollary 1 the product $f(x)(1 + g(x)/f(x))$ approaches infinity.

For the second statement, given any $M > 0$ we have $0 < |x - a| < \delta_1 \implies |f(x)| > M$. Let $\delta_2 > 0$ be such that in the deleted δ_2-neighborhood of a the functions $f(x)$ and $g(x)$ have the same sign. Then $0 < |x-a| < \min\{\delta_1, \delta_2\} \implies |f(x) + g(x)| > |f(x)| > M$. ∎

5.3 Examples

Example 22. $\lim_{x\to 0} \frac{1}{x} = \infty$.

Example 23. $\lim_{x\to 1} \frac{1}{x^2-1} = \infty$.

If x is close to 1, $\frac{1}{2} < x < \frac{3}{2}$, and so taking $\delta = \min\{\frac{1}{2}, \frac{1}{5M}\}$ we have $|f(x)| > M$ whenever $0 < |x-1| < \delta$. Note that both

$$\lim_{x\to 1} \frac{1}{x^2-1} = +\infty \text{ and } \lim_{x\to 1} \frac{1}{x^2-1} = -\infty$$

are *incorrect*! However we do have that

$$\lim_{x\to 1+} \frac{1}{x^2-1} = +\infty \text{ and } \lim_{x\to 1-} \frac{1}{x^2-1} = -\infty.$$

Example 24. For $a > 1$,

$$\lim_{x\to 0+} \log_a x = -\infty.$$

Example 25. $f(x) = \frac{1}{x}\sin\frac{1}{x}$, $x \to 0$.

Although the function is unbounded in every deleted neighborhood of the origin, it does *not* approach ∞ as $x \to 0$. To see this, we merely note that every deleted neighborhood of the origin contains infinitely many points of the form $1/(n\pi)$, $n \in \mathbb{Z}$, at which $f(x) = 0$. Hence, given any $M > 0$, there is no choice of δ such that $|f(x)| > M$ for *all* x satisfying $0 < |x| < \delta$.

Example 26. Consider the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{Q}, \\ -\frac{1}{x} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Here the limit at zero is ∞ with $\delta = \frac{1}{M}$.

Example 27. Consier the limits at zero of these fractions:

$$\frac{cx}{x}, \quad \frac{x}{x^2}, \quad \frac{x\sin\frac{1}{x}}{x}.$$

These show that whenever $f(x)$ and $g(x)$ approach 0 as $x \to a$, the ratio $f(x)/g(x)$ may or may not have limit, and if the limit exists it may be any finite number of infinity.

Consider now the limit $\lim_{x\to 0}[f(x) - g(x)]$ in the situations

$$f(x) = \frac{1}{x} + c, \quad g(x) = \frac{1}{x},$$
$$f(x) = \frac{2}{x}, \quad g(x) = \frac{1}{x},$$
$$f(x) = \frac{1}{x} + \sin\frac{1}{x}, \quad g(x) = \frac{1}{x}.$$

Example 28. Find the limits: (a) $\lim_{x\to\frac{\pi}{2}} \tan x$; (b) $\lim_{x\to\frac{\pi}{2}+} \tan x$; (c) $\lim_{x\to\frac{\pi}{2}-} \tan x$;

5.4 Problems

1. Formulate precise ε-δ definitions corresponding to each of the following formulas:

$$\lim_{x\to a+} f(x) = \infty, \quad \lim_{x\to a-} f(x) = \infty,$$
$$\lim_{x\to a+} f(x) = +\infty, \quad \lim_{x\to a-} f(x) = +\infty,$$
$$\lim_{x\to a+} f(x) = -\infty, \quad \lim_{x\to a-} f(x) = -\infty.$$

2. Let $f(x) = \dfrac{D(x)}{x}$, $x \neq 0$, where $D(x)$ is the Dirichlet function. Does $f(x)$ approach infinity as $x \to 0$?

3. Prove that if $f(x) \to \infty$ as $x \to a$ then $f(x)$ is unbounded in a deleted neighborhood of a. Is the converse true?

4. Is the expression $\infty + \dfrac{1}{\infty}$ indeterminate? How about $\infty/0$?

5. Prove that

$$\lim_{x\to a} f(x) = \infty$$

if and only if

$$\lim_{x\to a+} f(x) = \infty, \quad \lim_{x\to a-} f(x) = \infty.$$

Is the same true if ∞ replaced by $+\infty$ and $-\infty$?

6. Which of the following infinite limits equal ∞ (but not $+\infty$ or $-\infty$):

   (a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$;

   (b) $\lim\limits_{x\to 0} \dfrac{\sin x}{x^3}$;

   (c) $\lim\limits_{x\to 1} \dfrac{|\tan(x-1)|}{(x-1)^2}$;

   (d) $\lim\limits_{x\to 3} \dfrac{x+3}{x^2-9}$;

   (e) $\lim\limits_{x\to 3} \dfrac{(x+3)(-1)^{[x]}}{x^2-9}$;

   (f) $\lim\limits_{x\to 0} \dfrac{(-1)^{[x]}\sin x}{x^2}$?

7. Evaluate each of the following limits, thereby resolving an indeterminacy of the form $0 \cdot \infty$:

   (a) $\lim\limits_{x\to 0} x \cot 2x$;

   (b) $\lim\limits_{x\to \pi} \sin 2x \cot 2x$;

   (c) $\lim\limits_{x\to \frac{\pi}{4}} \left(\dfrac{\pi}{4} - x\right) \csc\left(\dfrac{3\pi}{4} + x\right)$.

8. Evaluate each of the following limits, thereby resolving an indeterminacy of the form $\infty - \infty$:

   (a) $\left(\dfrac{1}{\sin^2 x} - \dfrac{1}{4\sin^2 \frac{x}{2}}\right)$;

   (b) $\lim\limits_{x\to 0} (2\csc 2x - \cot x)$;

(c) $\lim_{x\to\frac{\pi}{2}} (\tan x - \sec x)$.

9. Prove that
$$\lim_{x\to 1} \frac{(-1)^{[x]}}{x-1} = -\infty.$$
What is the largest value of that δ such that $0 < |x-1| < \delta$ implies
$$\frac{(-1)^{[x]}}{x-1} < -1000?$$

10. Prove that

(a) $\lim_{x\to 0} \frac{\sin(x-1)}{x(x-1)^3} = \infty$;

(b) $\lim_{x\to 1} \frac{\sin(x-1)}{x(x-1)^3} = +\infty$;

(c) $\lim_{x\to 2} \frac{(-1)^{[x]+1}}{x^2-4} = -\infty$.

11. Evaluate

(a) $\lim_{x\to 0+} \frac{\sin(x-1)}{x(x-1)^3}$;

(b) $\lim_{x\to 0-} \frac{\sin(x-1)}{x(x-1)^3}$;

(c) $\lim_{x\to \frac{3\pi}{2}+} \sec x$;

(d) $\lim_{x\to 0-} \csc x$.

12. Evaluate ($n \in \mathbb{N}$)

(a) $\lim_{x\to (n+\frac{1}{2})\pi+} \tan x$;

(b) $\lim_{x\to (n+\frac{1}{2})\pi-} \tan x$;

(c) $\lim_{x\to n\pi+} \cot x$;

(d) $\lim_{x\to n\pi-} \cot x$.

13. Let $f(x)$ be function defined in a deleted neighborhood of the point a. Then $f(x)$ is said to approach L from the right as $x \to a$ (not to be confused with approaching L as $x \to a$ from the right) if, given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $0 < |x-a| < \delta$ implies $L < f(x) < L + \varepsilon$. This fact is expressed by writing $f(x) \to L+$ as $x \to a$ or
$$\lim_{x\to a} f(x) = L+.$$
Prove that $f(x) \to +\infty$ as $x \to a$ if and only if $1/f(x) \to 0+$ as $x \to a$.

14. Define
$$\lim_{x\to a} f(x) = L-$$
by analogy with the preceding problem. Prove that $f(x) \to -\infty$ as $x \to a$ if and only if $1/f(x) \to 0-$ as $x \to a$.

6 Lecture 5 – Limits at Infinity

6.1 Definitions

Unlike the previous sections, here we consider function defined on intervals of the form $(a, +\infty)$ and $(-\infty, a)$, $a \in \mathbb{R}$.

DEFINITION 17 (Limit at $+\infty$)**.** We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches $+\infty$,

and write

$$f(x) \to L \quad \text{as} \quad x \to +\infty,$$

or

$$f(x) \xrightarrow[x \to +\infty]{} L,$$

or

$$\lim_{x \to +\infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that $x > M$ implies $|f(x) - L| < \varepsilon$.

In logical symbolism this definition is written as

$$\lim_{(a,+\infty) \ni x \to +\infty} f(x) = L := \forall\ \varepsilon > 0\ \exists M > 0\ \forall x \in (a, +\infty)\ \ (x > M \Rightarrow |f(x) - L| < \varepsilon)\,.$$

The definition of limit at $-\infty$ is similar.

DEFINITION 18 (Limit at $-\infty$)**.** We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches $-\infty$,

and write

$$f(x) \to L \quad \text{as} \quad x \to -\infty,$$

or

$$f(x) \xrightarrow[x \to -\infty]{} L,$$

or

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that $x < -M$ implies $|f(x) - L| < \varepsilon$.

In logical symbolism this definition is written as

$$\lim_{(-\infty,a) \ni x \to -\infty} f(x) = L := \forall\ \varepsilon > 0\ \exists M > 0\ \forall x \in (-\infty, a)\ \ (x < -M \Rightarrow |f(x) - L| < \varepsilon)\,.$$

DEFINITION 19 (Limit at ∞)**.** We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches ∞,

and write

$$f(x) \to L \quad \text{as} \quad x \to \infty,$$

or

$$f(x) \xrightarrow[x \to \infty]{} L,$$

or

$$\lim_{x \to \infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that $|x| > M$ implies $|f(x) - L| < \varepsilon$.

In logical symbolism this definition is written as (here we assume $a > 0$)

$$\lim_{(-\infty,-a)\cup(a,+\infty)\ni x\to\infty} f(x) = L :\!= \forall\ \varepsilon > 0\ \exists M > 0\ \forall x \in (a, +\infty)\ \left(|x| > M \Rightarrow |f(x) - L| < \varepsilon\right).$$

NOTE 16. The choice of the letters M and δ in the definitions above is a matter of custom, M being a symbol for a typical large number and δ a symbol for a typical small number (δ gets smaller as M gets larger.) Thus the built-in connotation of the phrase "given any $M > 0$" is "given any $M > 0$, however large."

NOTE 17. One could also consider *infinite limits at infinity.* See Problem 1.

NOTE 18. All arithmetic operations for limits and all properties of limits we know from before apply for limits at infinity. State and prove them!

6.2 Theorems

Our only theorem here justifies change of variables in limits at infinity. With this theorem, any limit at infinity can be expressed as a limit at zero.

Theorem 18. The function $f(x) \to L$ as $x \to +\infty$ if and only if $f^*(t) \to L$ as $t \to 0+$, where $f^*(t) = f(1/t)$. (Note that here we allow the possibilities $L = +\infty$, $L = -\infty$, $L = \infty$.)

Proof. Suppose first that $L \in \mathbb{R}$. Then given $\varepsilon > 0$ there exists $M > 0$ such that $x > M$ implies $|f(x) - L| < \varepsilon$. Then taking $\delta = 1/M$ we see that $0 < t < \delta$ implies $|f^*(t) - L| < \varepsilon$.

Suppose now that $L = +\infty$. Then for every $M > 0$ there exists $M' > 0$ such that $x > M'$ implies $f(x) > M$. Then taking $\delta = 1/M'$ we see that $0 < t < \delta$ implies $f^*(t) > M$.

The case $L = -\infty$ is dealt with in a similar way. The converse in all three cases is obtained by reversing the argument. ■

6.3 Examples

Example 29. For $a > 1$,

$$\lim_{x\to+\infty} a^x = +\infty \quad \text{and} \quad \lim_{x\to-\infty} a^x = 0.$$

Analogously, for $0 < a < 1$,

$$\lim_{x\to+\infty} a^x = 0 \quad \text{and} \quad \lim_{x\to-\infty} a^x = +\infty.$$

Example 30. For $a > 1$,

$$\lim_{x\to+\infty} \log_a x = +\infty.$$

(Recall that we already established that

$$\lim_{x\to0+} \log_a x = -\infty.)$$

Example 31. Limits at infinity of polynomial and rational functions. (Here the leading coefficients are assumed nonzero.)

$$\lim_{x\to\pm\infty} (a_n x^n + \cdots + a_1 x + a_0) = \begin{cases} \begin{cases} +\infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases} & \text{if } n \text{ is even} \\ \begin{cases} \pm\infty, & \text{if } a_n > 0 \\ \mp\infty, & \text{if } a_n < 0 \end{cases} & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{x\to\infty} \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0} = \begin{cases} \dfrac{a_m}{b_m}, & \text{if } m = n \\ 0, & \text{if } m < n \\ \infty, & \text{if } m > n \end{cases}$$

Example 32. Evaluate

$$\lim_{x\to+\infty} \left(\sqrt{x^2 + x - 1} - \sqrt{x^2 - x + 1}\right).$$

What if $x \to -\infty$?

6.4 Problems

1. Formulate precise ε-δ definitions corresponding to each of the following formulas:

$$\lim_{x\to+\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = \infty,$$
$$\lim_{x\to+\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = +\infty,$$
$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Also write these definitions using logical symbolism.

2. Prove that

$$\lim_{x\to+\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right) = 0.$$

3. Evaluate

(a) $\lim\limits_{x\to+\infty} \left(\dfrac{5}{3x^3} - \dfrac{81}{9-\sqrt{x}}\right)$;

(b) $\lim\limits_{x\to-\infty} \left(2 + \dfrac{1}{\sqrt{1-x}} - \dfrac{2}{x^3}\right)$;

(c) $\lim\limits_{x\to\pm\infty} \dfrac{2x^2-5x+4}{5x^2-2x-3}$.

4. Evaluate

(a) $\lim\limits_{x\to+\infty} \dfrac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}$;

(b) $\lim\limits_{x\to-\infty} \dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}}$;

(c) $\lim\limits_{x\to\pm\infty} \dfrac{(x+1)(x^2+1)\cdots(x^n+1)}{[(nx)^n+1]^{\frac{n+1}{2}}}$.

5. Evaluate

(a) $\lim\limits_{x\to+\infty} \left(\sqrt{x^2+3x} - x\right)$;

(b) $\lim\limits_{x\to+\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)$;

(c) $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+1} - \sqrt{x^2-4x}\right)$;

(d) $\lim\limits_{x\to+\infty} \left(x - \sqrt{x^2-x+1}\right)$;

(e) $\lim\limits_{x\to+\infty} \left(x - \sqrt{x^2-a^2}\right)$;

(f) $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2-ax}\right)$.

6. (a) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$;

(b) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}}$;

(c) $\lim\limits_{x\to+\infty} \left(\sqrt{(x+a)(x+b)} - x\right)$;

(d) $\lim\limits_{x\to+\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right)$.

7 Lecture 6 — Limits of Sequences

Here we consider a very special type of functions whose domain is the set $\mathbb{N}$ of natural numbers. These are called *sequences*. Having defined the limit of a sequence we can redefine the concept of limit for a function. As we will see that will be very fruitful.

7.1 Definition

DEFINITION 20 (Sequence)**.** A function $f\colon \mathbb{N} \to X$ whose domain of definition is the set of natural numbers is called a *sequence*.

NOTE 19. The values $f(n)$ of the function f are called the *terms of the sequence*. It is customary to denote them by a symbol for an element of the set into which the mapping goes, endowing each symbol with the corresponding index of the argument. Thus, $x_n := f(n)$. In this connection the sequence itself is denoted $\{x_n\}$, and also written as $x_1, x_2, \dots, x_n$. It is called a *sequence in* X or a *sequence of elements of* X.

The element x_n is called the *nth term of the sequence* or *the general term*. The curly brackets emphasize the distinction between the sequence $\{x_n\}$ and its general term.

Throughout the course we shall be considering only sequences $f\colon \mathbb{N} \to \mathbb{R}$ of real numbers.

DEFINITION 21 (ε-δ of limit of sequence)**.** We say that

“$\{x_n\}$ approaches (or tends to) $L \in \mathbb{R}$ as n approaches (or tends to) ∞”

or

“$\{x_n\}$ has limit L as n tends to ∞”,

and write

$$x_n \to L \quad \text{as} \quad n \to \infty,$$

or

$$x_n \xrightarrow[n\to\infty]{} L,$$

or

$$\lim_{n\to\infty} x_n = L$$

if given any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - L| < \varepsilon$.

A sequence having a limit is said to be *convergent*. A sequence that does not have a limit is said to be *divergent*.

We say that

“$\{x_n\}$ approaches (or tends to) ∞ as n approaches (or tends to) ∞”

or

“$\{x_n\}$ is *infinitely large*”,

and write

$$x_n \to \infty \quad \text{as} \quad n \to \infty,$$

or

$$x_n \xrightarrow[n\to\infty]{} \infty,$$

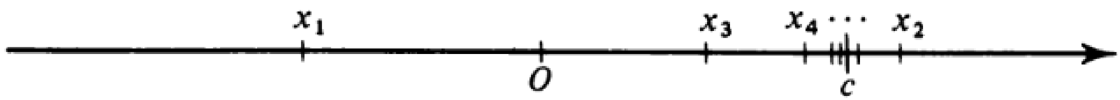
or

$$\lim_{n\to\infty} x_n = \infty$$

if given any $M > 0$ there exists a $N \in \mathbb{N}$ such that $n > N$ implies $|x_n| > M$.

NOTE 20. Since n is inherently positive, $n \to \infty$ can only mean $n \to +\infty$.

NOTE 21. Definition 28 has the following geometric meaning: If $x_n \to c \neq \infty$ as $n \to \infty$, then every neighborhood of L contains all but but a finite number of terms of $\{x_n\}$, in fact all terms of $\{x_n\}$ starting from some value of n. This fact is easy to express graphically.



In logical symbolism this definition is written as

$$\lim_{n\to\infty} x_n = L := \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \left(n > N \Rightarrow |x_n - L| < \varepsilon\right),$$

or

$$\lim_{n\to\infty} x_n = \infty := \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \left(n > N \Rightarrow |x_n| > M\right).$$

DEFINITION 22. A convergent sequence whose limit is zero is called *infinitely small* or *infinitesimal*.

7.2 Theorems

All of the major theorems about limits functions apply for sequences and we shall not prove them here. These included theorems about arithmetic operations with limits, limit of a composition, as well as the Squeeze theorem.

DEFINITION 23. Let $n = n(k) = n_k$ be a *strictly monotone increasing* function defined on $\mathbb{N}$. A *subsequence* of a sequence $\{x_n\}_{n=1}^{\infty}$ is any sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$, where n_k is as described.

NOTE 22. In layman's terms, a subsequence is obtained from a given sequence by "erasing" some of its terms, but rearranging of the terms is not allowed.

Example 33. Consider the sequence $x_n = (-1)^n$. By letting $n(k) := 2k$ we get the subsequence $x_n = x_{2k} = (-1)^{2k} = 1$; and by letting $n(k) := 2k - 1$ we get the subsequence $x_n = x_{2k-1} = (-1)^{2k-1} = -1$. Note that both subsequences are convergent even though the original sequence is not.

Theorem 19. If the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a (finite or infinite) then every subsequence of $\{x_n\}$ also converges to the same limit a.

Proof. Let a (finite for simplicity) be the limit of $\{x_n\}_{n=1}^{\infty}$ and let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence. Fix an arbitrary $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ as soon as $n > N$. Since $n_k = n(k)$ is a strictly monotone increasing function of k, we have $n_k \to \infty$ as $k \to \infty$. Thus there exists $K \in \mathbb{N}$ such that $n_k > N$ as soon as $k > K$. Therefore we have

$$k > K \Longrightarrow n_k > N \Longrightarrow |x_{n_k} - L| < \varepsilon.$$

The case when $a = \infty$ is left as an exercise. ∎

NOTE 23. This theorem gives a convenient way of proving that a sequence is *divergent*. To prove divergence one has to exhibit two subsequence having different limits (finite or infinite).

7.3 Examples

Example 34. Consider the sequences

$$x_n = \frac{1}{n}, \quad y_n = -\frac{1}{n}, \quad z_n = \frac{(-1)^n}{n}, \quad n \in \mathbb{N};$$

all three are convergent, with limit 0, that is, all three are infinitesimal.

Example 35. The sequences $x_n = n$, $y_n = -n$, and $z_n = (-1)^n n$ are infinitely large.

Example 36. The sequence $x_n = (-1)^n$ is divergent.

NOTE 24. Theorems that govern arithmetic operations with limits, passage to limits in inequalities, etc. all remain true for sequences. In fact, many treatments of Calculus, such as Zorich, begin with sequences and talk about functions later. *Consider writing the notes using this approach!*

Example 37. $\lim\limits_{n\to\infty} \left[(n+1)^k - n^k\right] = 0$ for $0 < k < 1$. This is clear since

$$0 < \left[(n+1)^k - n^k\right] = n^k \left[\left(1+\frac{1}{n}\right)^k - 1\right] < n^k \left[\left(1+\frac{1}{n}\right) - 1\right] = n^{k-1} = \frac{1}{n^{1-k}} \xrightarrow[n\to\infty]{} 0.$$

Example 38. Compute the limit $\lim\limits_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$.

Example 39. Evaluate $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2}\sin n!}{n+1}$.

Example 40. Evaluate $\lim\limits_{n\to\infty} q^n$, $q \in \mathbb{R}$.

If $0 < |q| < 1$, then $n > \log_{|q|} \varepsilon$ implies $|q^n| < \varepsilon$. The situations $q = 0$ and $q = 1$ are trivial. If $q = -1$, see above. If $|q| > 1$, then for any $M > 0$, $n > \log_q M$ implies $|q^n| > M$. Thus if $|q| < 1$, this limit is zero.

Example 41. Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} q^k$, $q \in \mathbb{R}$.

Let $S_n := \sum\limits_{k=1}^{n} q^k$, so $S_n - qS_n = q - q^{n+1}$ and $S_n = \dfrac{q-q^{n+1}}{1-q}$, $q \neq 1$. (If $q = 1$ then $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} q^k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} 1 = \lim\limits_{n\to\infty} n = \infty$.) If $|q| < 1$, we have $\lim\limits_{n\to\infty} S_n = \dfrac{q}{1-q}$.

Example 42. $\forall a \in \mathbb{R}$, $a > 0$, $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.

Let $x_n := \sqrt[n]{a} - 1$ so $a = (1 + x_n)^n$. If $a > 1$ we have $x_n > 0$ for all $n \in \mathbb{N}$ and then

$$a = (1+x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n > nx_n,$$

from which it follows that $0 < x_n < \frac{a}{n}$. By the Squeeze Principle $\lim_{n\to\infty} x_n = 0$, and thus $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

The case $a = 1$ is trivial; for $0 < a < 1$ we have

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{1/a}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1/a}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \sqrt[n]{1/a}} = \frac{1}{1} = 1.$$

Example 43. $\lim\limits_{n\to\infty} \dfrac{a^n}{n^k} = +\infty$, for any $a > 1$, $k > 0$. Consider

$$a^n = (1 + (a-1))^n = 1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \cdots > \frac{n(n-1)}{2}(a-1)^2.$$

If $n > 2$, we have $n - 1 > \frac{n}{2}$ and so

$$a^n > \frac{n^2(a-1)^2}{4}.$$

For $k = 1$ we have

$$\frac{a^n}{n} > \frac{n(a-1)^2}{4} \Rightarrow \lim_{n\to\infty} \frac{a^n}{n} = +\infty.$$

Note that this is true for *any* $a > 1$. Therefore, if $k > 1$, for sufficiently large n,

$$\frac{a^n}{n^k} = \left(\frac{(a^{1/k})^n}{n}\right)^k > \frac{(a^{1/k})^n}{n} \quad \text{since} \quad \frac{(a^{1/k})^n}{n} > 1 \quad \text{for sufficiently large } n.$$

Hence $\lim_{n\to\infty} \frac{a^n}{n^k} = +\infty$ if $k > 1$. For $0 < k < 1$, we have

$$\frac{a^n}{n^k} > \frac{a^n}{n},$$

and thus $\lim_{n\to\infty} \frac{a^n}{n^k} = +\infty$ for all $k > 0$.

Example 44. $\lim_{n\to\infty} \sqrt[n]{n} = 1$. We again use the inequality

$$a^n > \frac{n^2(a-1)^2}{4} \text{ for all } a > 1.$$

Setting $a = \sqrt[n]{n} > 1$ (provided that $n > 1$) we have

$$n > \frac{(\sqrt[n]{n}-1)^2 n^2}{4} \Rightarrow 0 < \sqrt[n]{n} - 1 < \frac{2}{\sqrt{n}}.$$

Thus $\lim_{n\to\infty} (\sqrt[n]{n} - 1) = 0$, and $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Example 45. $\lim_{n\to\infty} \frac{\log_a n}{n} = 0$, for any $a > 1$. For any $\varepsilon > 0$ we have $a^\varepsilon > 1$ and so for sufficiently large $n \in \mathbb{N}$, $\sqrt[n]{n} < a^\varepsilon$ since $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Taking logarithm base a of both sides we get $\lim_{n\to\infty} \frac{\log_a n}{n} = 0$.

Example 46. Let $a_1, \dots, a_k$ be positive numbers. Then

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \dots + a_k^n} = \max\{a_1, \dots, a_k\}.$$

This follows immediately from the estimates

$$M \leqslant \sqrt[n]{a_1^n + \dots + a_k^n} \leqslant \sqrt[n]{k} \cdot M,$$

where M is the maximum of the a_i, $1 \leqslant i \leqslant k$.

7.4 Limits of Monotonic Sequences

DEFINITION 24. A sequence $\{x_n\}$ is called *monotonic* if for all $n \in \mathbb{N}$ there holds either $x_{n+1} \geqslant x_n$ (in which the sequence is called *nondecreasing*) or $x_{n+1} \leqslant x_n$ (in which case the sequence is called nonincreasing.)

DEFINITION 25. A sequence $\{x_n\}$ is called *bounded* if the set of numbers $\{x_n \mid n \in \mathbb{N}\}$ is bounded as a subset of $\mathbb{R}$. That is, there exists an $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

Theorem 20. Let $\{x_n\}$ be a monotonic nondecreasing sequence. If $\{x_n\}$ is bounded from above then $\{x_n\}$ has a finite limit. Otherwise, $x_n \xrightarrow[n\to\infty]{} \infty$.

Proof. Assume first that $\{x_n\}$ is bounded from above. Recall that the Completeness Axiom of the real numbers implies that every nonempty set of real numbers bounded from above has a *unique supremum.* Let a be the supremum of $\{x_n\}$. We claim that $\lim\limits_{n\to\infty} x_n = a$. By definition of supremum

- $\forall n \in \mathbb{N}$, $x_n \leqslant a$, and
- $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ with $x_N > a - \varepsilon$.

Since $\{x_n\}$ is nondecreasing, for all $n > N$ we have $a \geqslant x_n > a - \varepsilon$, and so $n > N$ implies $|x_n - a| < \varepsilon$.

If $\{x_n\}$ is not bounded from above then for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_N > M$ and therefore (since $\{x_n\}$ is nondecreasing) $n > N$ implies $x_n > M$. That is, $x_n \xrightarrow[n\to\infty]{} \infty$. ■

NOTE 25. The case of a monotone nonincreasing sequence bounded from below is dealt with the same way (see Problem 7.)

7.5 Examples of Monotonic Sequences

Example 47. We show that the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and is finite. We then set

$$\boxed{\mathrm{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.} \tag{5}$$

Consider first the sequence $x_n = \left(1+\frac{1}{n}\right)^{n+1}$. Let us first of all show that $\{x_n\}$ is decreasing. We will need *Bernoulli's Inequality* (see Lecture 0): for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}$ with $x > -1$,

$$(1+x)^n \geqslant 1+nx.$$

Now (we indicate by B where we use Bernoulli's Inequality), for every $n \in \mathbb{N}$,

$$\begin{aligned}\frac{x_n}{x_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+1}\cdot\frac{1}{1+\frac{1}{n+1}} = \left(1+\frac{1}{n(n+2)}\right)^{n+1}\cdot\frac{1}{1+\frac{1}{n+1}}\\ &\overset{B}{\geqslant} \left(1+\frac{n+1}{n(n+2)}\right)\cdot\frac{1}{1+\frac{1}{n+1}} > \left(1+\frac{n+1}{n(n+2)+1}\right)\cdot\frac{1}{1+\frac{1}{n+1}} \geqslant \left(1+\frac{1}{n+1}\right)\cdot\frac{1}{1+\frac{1}{n+1}} = 1.\end{aligned}$$

Therefore $x_{n+1} < x_n$ for all $n \in \mathbb{N}$, and so $\{x_n\}$ is indeed monotone decreasing. Since obviously $x_n > 0$ for all $n \in \mathbb{N}$, $\{x_n\}$ is also bounded from below. By Theorem 20, the sequence $\{x_n\}$ has a limit. We call this limit by e:

$$\mathrm{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}.$$

Now

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)\right] = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Finally we note that (again using Bernoulli's Inequality)

$$x_5 = \left(1+\frac{1}{5}\right)^6 < 3, \text{ and } x_n = \left(1+\frac{1}{n}\right)^{n+1} \overset{B}{\geqslant} 1+\frac{n+1}{n} > 2,$$

so $2 < \mathrm{e} < 3$.

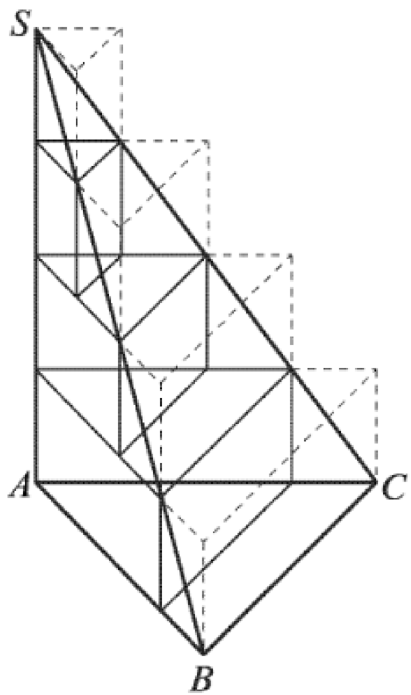
7.6 Problems

1. Show that a monotone nonincreasing sequence has a finite limit if bounded from below and has limit $-\infty$ otherwise.

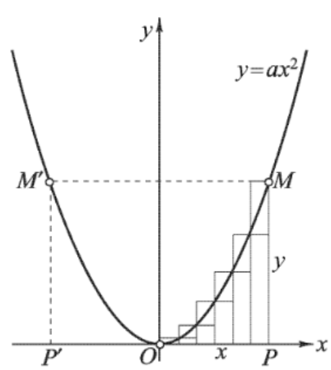
2. Prove Theorem 19 for the case $a = \infty$.

3. Compute the volume of the pyramid of $SABC$ shown in the figure under the assumption that you know the area of $\triangle ABC$ and the height AS. (The picture should also serve as a hint on how to do that.) You may need to use the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \in \mathbb{N}. \tag{6}$$



4. Compute the area of the figure OPM formed by the portion OM of the parabola $y = ax^2$ $(a > 0)$, the segment OP of the x-axis and segment PM. (The picture should also serve as a hint on how to do that.) You may need to use formula 6.



5. Show that the sequence $x_n := \left(1+\frac{1}{n}\right)^n$ is monotone increasing and is bounded above by 3. This gives another proof that x_n is increasing. **Hint:** Use the binomial theorem.

6. This problem explores a way to approximate e. Our argument will imply that e is *irrational*. **FINISH!!!**

7. Show that a sequence that is nonincreasing and bounded from below is convergent.

8 Lecture 7 — Another Look at Limits of Functions

The point of this lecture is to give a completely different definition of limit at a point using sequences. This approach is due to Heine. Here we as always assume that $a \in \mathbb{R} \cup \{\infty\}$ is a limit point of the domain D of a function f (the novelty here is that we allow $a = \infty$).

Recall our old definition of limit at a point and at infinity.

DEFINITION 26 (ε-δ definition of limit)**.** Let $a \in \mathbb{R}$ be a limit point of $D = \mathrm{Dom}(f)$. We say that

“f approaches (or tends to) $L \in \mathbb{R}$ as x approaches (or tends to) a”

or

“f has limit L as x tends to a”,

and write

$$f(x) \to L \quad \text{as} \quad x \to a,$$

or

$$f(x) \xrightarrow[x\to a]{} L,$$

or

$$\lim_{x\to a} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 < |x - a| < \delta$, there holds $|f(x) - L| < \varepsilon$.

DEFINITION 27 (Limit at ∞)**.** We say that

“f approaches (or tends to) $L \in \mathbb{R}$ as x approaches ∞,

and write

$$f(x) \to L \quad \text{as} \quad x \to \infty,$$

or

$$f(x) \xrightarrow[x\to \infty]{} L,$$

or

$$\lim_{x\to \infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that $|x| > M$ implies $|f(x) - L| < \varepsilon$.

Now recall the definition of limit of a sequence.

DEFINITION 28 (ε-δ of limit of sequence)**.** We say that

“$\{x_n\}$ approaches (or tends to) $L \in \mathbb{R}$ as n approaches (or tends to) ∞”

or

“$\{x_n\}$ has limit L as n tends to ∞”, and write

$$x_n \to L \quad \text{as} \quad n \to \infty,$$

or

$$x_n \xrightarrow[n\to\infty]{} L,$$

or

$$\lim_{n\to\infty} x_n = L$$

if given any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - L| < \varepsilon$.

A sequence having a limit is said to be *convergent*. A sequence that does not have a limit is said to be *divergent*.

We say that

"$\{x_n\}$ approaches (or tends to) ∞ as n approaches (or tends to) ∞"

or

"$\{x_n\}$ is *infinitely large*",

and write

$$x_n \to \infty \quad \text{as} \quad n \to \infty,$$

or

$$x_n \xrightarrow[n\to\infty]{} \infty,$$

or

$$\lim_{n\to\infty} x_n = \infty$$

if given any $M > 0$ there exists a $N \in \mathbb{N}$ such that $n > N$ implies $|x_n| > M$.

In logical symbolism this definition is written as (here we assume $a > 0$)

$$\lim_{n\to\infty} x_n = L := \forall\ \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \left(n > N \Rightarrow |x_n - L| < \varepsilon\right).$$

We now give a different definition of limit due to Heine.

DEFINITION 29 (Using sequences)**.** Let $a \in \mathbb{R}$ be a limit point of $D = \mathrm{Dom}(f)$. We say that

"f approaches (or tends to) $L \in \mathbb{R}$ as x approaches (or tends to) a"

or

"f has limit L as x tends to a",

and write

$$f(x) \to L \quad \text{as} \quad x \to a,$$

or

$$f(x) \xrightarrow[x\to a]{} L,$$

or

$$\lim_{x\to a} f(x) = L$$

if given any sequence $\{x_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} x_n = a, \quad \forall n \in \mathbb{N}\ x_n \in D,$$

we have

$$\lim_{n\to\infty} f(x_n) = L.$$

First of all we prove that definitions 26 and 29 are logically equivalent. That is, if L is the limit of $f(x)$ at a in the sense of one definition, L is the limit of $f(x)$ at a in the sense of the other definition.

Theorem 21. Definition 26 is logically equivalent to Definition 29. (the cases $L = \infty$ and $a = \infty$ are allowed.)

Proof. Assume for simplicity that both a and L are in $\mathbb{R}$.

Suppose first that $f(x) \xrightarrow[x \to a]{} L$ in the sense of Definition 26. Then fixing an arbitrary $\varepsilon > 0$ we find $\delta > 0$ such that for all $x \in D$ satisfying $0 < |x - a| < \delta$ there holds $|f(x) - L| < \varepsilon$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of points in D with limit a. Then for the δ just found there exists $N \in \mathbb{N}$ for which $|x_n - a| < \delta$ as soon as $n > N$. Thus we have

$$n > N \Longrightarrow |x_n - a| < \delta \Longrightarrow |f(x_n) - L| < \varepsilon.$$

This implies that $f(x_n) \xrightarrow[n \to \infty]{} L$. We have proved that for $f(x)$ having limit L implies that $f(x)$ converges to L on any sequence converging to a.

Suppose conversely that for *any* sequence $\{x_n\}_{n=1}^{\infty}$ converging to a, the sequence $f(x_n)$ converges to L. Suppose by way of contradiction that $\lim_{x \to a} f(x) \neq L$. That is, there exists $\varepsilon > 0$ such that for whatever $\delta > 0$ we choose, there must exist an $x \in \mathrm{Dom}(f)$ with the property that $0 < |x - a| < \delta$ and yet $|f(x) - L| \geqslant \varepsilon$. By taking $\delta = \delta_n = \frac{1}{n}$ we construct a sequence $\{x_n\}_{n=1}^{\infty}$ for which

$$0 < |x_n - a| < \frac{1}{n} \ \wedge \ |f(x_n) - L| \geqslant \varepsilon.$$

Then clearly $\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} f(x_n) \neq L$ contradicting the premise. ■

NOTE 26. It is enough to assume only the *existence* of limit for every sequence $\{f(x_n)\}_{n=1}^{\infty}$ corresponding to a sequence $\{x_n\}_{n=1}^{\infty}$ converging to a for us to claim that all such sequences $\{f(x_n)\}_{n=1}^{\infty}$ have the *same limit*. To see this suppose by way of contradiction that for some two sequences $\{x'_n\}_{n=1}^{\infty}$ and $\{x''_n\}_{n=1}^{\infty}$ converging to a we had

$$f(x'_n) \xrightarrow[n \to \infty]{} L' \text{ and } f(x''_n) \xrightarrow[n \to \infty]{} L''$$

with $L' \neq L''$. Then the sequence

$$x'_1, x''_1, x'_2, x''_2, \dots, x'_n, x''_n, \dots$$

converges to a (why?!), yet the corresponding sequence

$$f(x'_1), f(x''_1), f(x'_2), f(x''_2), \dots, f(x'_n), f(x''_n), \dots$$

has no limit (why?!).

8.1 Examples

1. Our most important result here will be that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{t\to 0}(1+t)^{\frac{1}{t}} = \mathrm{e}.$$

Note that this is an indeterminacy of the form 1^∞. Let n_k be any sequence with limit $+\infty$. Then since $\left(1+\frac{1}{n_k}\right)^{n_k}$ is a subsequence of the sequence $\left(1+\frac{1}{n}\right)^n$, we have

$$\left(1+\frac{1}{n_k}\right)^{n_k} = \mathrm{e}.$$

Suppose now that x runs over a sequence $\{x_k\}_{k=1}^\infty$ having limit $+\infty$. We may assume that $x_k > 1$. Now let $n_k = \lfloor x_k \rfloor$ so

$$n_k \leqslant x_k < n_k + 1 \Longrightarrow \frac{1}{n_k+1} < \frac{1}{x_k} \leqslant \frac{1}{n_k}.$$

It follows that

$$\left(1+\frac{1}{n_k+1}\right)^{n_k} \leqslant \left(1+\frac{1}{x_k}\right)^{x_k} \leqslant \left(1+\frac{1}{n_k}\right)^{n_k+1}.$$

Now

$$\left(1+\frac{1}{n_k+1}\right)^{n_k} = \frac{\left(1+\frac{1}{n_k+1}\right)^{n_k+1}}{1+\frac{1}{n_k+1}} \xrightarrow[k\to\infty]{} \frac{\mathrm{e}}{1} = \mathrm{e},$$

$$\left(1+\frac{1}{n_k}\right)^{n_k+1} = \left(1+\frac{1}{n_k}\right)^{n_k} \cdot \left(1+\frac{1}{n_k}\right) \xrightarrow[k\to\infty]{} \mathrm{e}\cdot 1 = \mathrm{e}.$$

Let us now assume that $x_k \to -\infty$ (so we can assume that all $x_k < -1$). If we set $x_k = -y_k$ then $y_k \to +\infty$. Now

$$\left(1+\frac{1}{x_k}\right)^{x_k} = \left(1-\frac{1}{y_k}\right)^{-y_k} = \left(\frac{y_k}{y_k-1}\right)^{y_k} = \left(1+\frac{1}{y_k-1}\right)^{y_k}$$
$$= \left(1+\frac{1}{y_k-1}\right)^{y_k-1}\cdot\left(1+\frac{1}{y_k-1}\right) \xrightarrow[k\to\infty]{} \mathrm{e}\cdot 1 = \mathrm{e}.$$

If a parameter t runs over a sequence of positive or negative numbers converging to zero then $x\frac{1}{t}$ has infinite limit. Then

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}.$$

8.2 Problems

1. Prove Theorem 28 in the other three cases: 1) $a \in \mathbb{R}$, $L = \infty$, 2) $a = \infty$, $L \in \mathbb{R}$, and 3) $a = L = \infty$.

9 Lecture 8 — Continuity (Part I)

Let f be a real-valued function defined in a neighborhood of a point $a \in \mathbb{R}$. In intuitive terms the function f is continuous at a if its value $f(x)$ approaches the value $f(a)$ that it assumes at the point a itself as x gets nearer to a.

We shall now make this description of the concept of continuity of a function at a point precise.

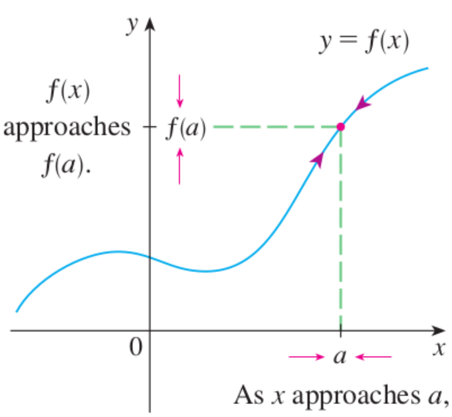
9.1 Definition and Examples

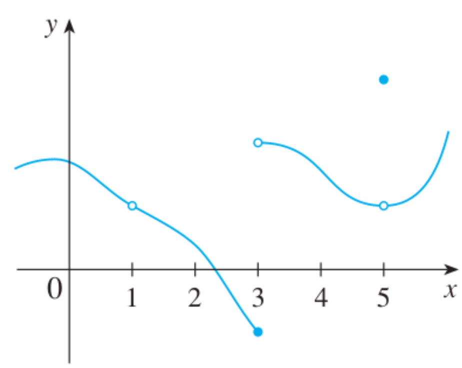
DEFINITION 30 (ε-δ definition of continuity at a point)**.** Let $a \in D = \mathrm{Dom}(f)$ be a limit point of D. We say that

"f is *continuous at* a"

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 \leqslant |x - a| < \delta$, there holds $|f(x) - f(a)| < \varepsilon$.

NOTE 27. Note that this definition is very similar to the definition of limit at point a. In fact, the only significant difference is that in the definition of limit we require $0 < |x - a| < \delta$ whereas in the definition of continuity we have $0 \leqslant |x - a| < \delta$. This motivates the following more practical definition of continuity at a point.





In logical symbolism this definition is written as

$$f \text{ is continuous at } a := \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D \ \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \right).$$

DEFINITION 31 (Definition of continuity at a point through limit)**.** Let $a \in D = \mathrm{Dom}(f)$ be a limit point of D. We say that

"f is *continuous at* a"

if f has a (finite) limit at a and

$$\lim_{x \to a} f(x) = f(a).$$

Theorem 22. Definition 30 is equivalent to Definition 31, that is, if f is continuous at a point a in the sense of one of the definitions then it also continuous in the sense of the other definition.

Proof. Suppose f is continuous at a point a in the sense of Definition 30. Then for every positive $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 \leqslant |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

Then obviously we have

$$0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

In other words, the number $f(a)$ (which exists since f is defined at a) satisfies the definition of limit of f at a. Thus $\lim_{x\to a} f(x) = f(a)$, and f is continuous at a in the sense of Definition 31.

Suppose conversely that f is continuous at a in the sense of Definition 31, that is $\lim_{x\to a} f(x) = f(a)$. Then by definition of limit for every positive $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

Since clearly $|f(a) - f(a)| = 0 < \varepsilon$ we may include the case $x = a$, so

$$0 \leqslant |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon,$$

and f is continuous at a in the sense of Definition 30. ■

DEFINITION 32. We say that a function $f\colon D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous on D* if f is continuous at every point of D.

Example 48. We know from the past that

$$\forall a \in \mathbb{R}, \quad \lim_{x\to a} \sin x = \sin a,$$

$$\forall a \in [0, +\infty), \quad \lim_{x\to a} \sqrt{x} = \sqrt{a},$$

so these are examples of functions that are continuous at every point of their domains. That is, $\sin x$ is continuous on all of $\mathbb{R}$ and $\sqrt{x}$ is continuous on $[0, +\infty)$.

Example 49. The function

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

is defined and continuous on all of $\mathbb{R}$. *Why?*

DEFINITION 33. If the function $f\colon D \to \mathbb{R}$ is not continuous at a point of D, this point is called a *point of discontinuity* or simply a discontinuity of f.

Example 50. The function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

is defined on all of $\mathbb{R}$. It is however *not* continuous at $x = 0$. On the other hand it is continuous at every point of $\mathbb{R} \setminus \{0\}$, that is, at every $x \neq 0$.

Example 51. The Dirichlet function

$$D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is defined on all of $\mathbb{R}$. It is however discontinuous on all of $\mathbb{R}$ simply because there is no limit at any point of $\mathbb{R}$.

Example 52. Consider the function

$$R(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q}, \text{ where } \frac{m}{n} \text{ is in lowest terms}, n \in \mathbb{N}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We claim that any point $a \in \mathbb{R} \setminus \mathbb{Q}$ we have $\lim_{x\to a} R(x) = 0 = R(a)$, and so R is continuous at a. Let $\varepsilon > 0$ be arbitrary and let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon$. Then any (bounded) neighborhood of a contains only finitely many rational numbers $\frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $n < N$. Thus by taking $\delta > 0$ small enough we may assume that the deleted δ-neighborhood of a contains no such points at all. In other words, for any rational point $\frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, in $(a - \delta) \cup (a + \delta)$ we have $n > N$. Then for any $x \in (a - \delta) \cup (a + \delta)$ we either have $R(x) = 0$ or $R(x) = \frac{1}{n} < \frac{1}{N} < \varepsilon$. In either case $R(x) = |R(x)| < \varepsilon$ as long as $|x - a| < \delta$. Note that the above argument applies to any point $a \in \mathbb{R}$. If $a \in \mathbb{Q} \setminus \{0\}$ then $R(a) \neq 0$, and so any such point is a point of discontinuity.

We will have more examples of continuous and discontinuous functions soon.

9.2 Continuify from the Right and Continutiy from the Left

By analogy with one-sided limits we can define "one-sided continuity".

DEFINITION 34 (ε-δ definition of right continuity at a point)**.** Let $a \in D = \mathrm{Dom}(f)$ be a limit point of D. We say that

"f is *right-continuous at a*"

or

"f is *continuous at a from the right*"

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $0 \leqslant x - a < \delta$, there holds $|f(x) - f(a)| < \varepsilon$. Equivalently, we say that

"f is *continuous at a from the right*"

if

$$\lim_{x\to a+} f(x) = f(a).$$

DEFINITION 35 (ε-δ definition of left continuity at a point)**.** Let $a \in D = \mathrm{Dom}(f)$ be a limit point of D. We say that

"f is *left-continuous at a*"

or

"f is *continuous at a from the left*"

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in D$ satisfies $-\delta < x - a \leqslant 0$, there holds $|f(x) - f(a)| < \varepsilon$. Equivalently, we say that

"f is *continuous at a from the left*"

if

$$\lim_{x\to a-} f(x) = f(a).$$

Recall that a limit a point exists if and only if both one-sided limit exist and are equal. Hence

Theorem 23. The function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $a \in D$ if and only if f is both left-continuous at a and left-continuous at a.

Example 53. Consider the following three functions:

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0, \\ -1, & \text{if } x = 0, \end{cases} \qquad h(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $f(x)$ is right-continuous but not left-continuous at $x = 0$; $g(x)$ is left-continuous but not right-continuous at $x = 0$; $h(x)$ is neither left-continuous nor right-continuous at $x = 0$.

9.3 Points of Discontinuity

Example 54. The function

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

has the limit $\lim_{x\to 0} f(x) = 1$, but $f(0) = 0 \neq 1$, and therefore 0 is a point of discontinuity.

We remark, however, that in this case, if we were to change the value of the function at the point 0 and set it equal to 1 there, we would obtain a function that is continuous at 0, that is, we would remove the discontinuity.

DEFINITION 36 (Removable discontinuity)**.** If a point of discontinuity $a \in D$ of the function $f\colon D \to \mathbb{R}$ is such that limit $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$ then a is called a *removable discontinuity* of the function f.

Clearly the function

$$f_1(x) = \begin{cases} f(x), & \text{for } x \in D,\, x \neq a, \\ f(a), & \text{for } x = a \end{cases}$$

is continuous on at a and is equal to f at all other points.

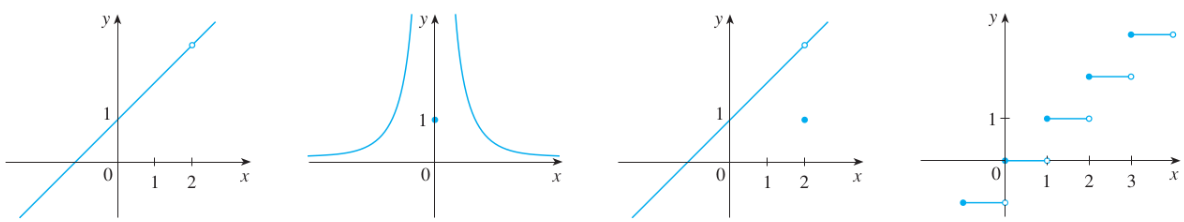
DEFINITION 37 (Discontinuity of first kind)**.** The point $a \in D$ is called a discontinuity *of first kind* for the function $f\colon D \to \mathbb{R}$ if the following limits[3] exist:

$$f(a-0) = \lim_{x\to a-} f(x), \qquad f(a+0) = \lim_{x\to a=} f(x),$$

but at least one of them is not equal to the value $f(a)$ that the function assumes at a.

DEFINITION 38 (Discontinuity of second kind)**.** If $a \in D$ is a point of discontinuity of the function $f\colon D \to \mathbb{R}$ and at least one of the two limits in Definition 37 does not exist then a is called a discontinuity of second kind.

Example 55. Determine the type of discontinuity at each point.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

Go back to Examples 50, 51, 52, and 53 and determine what types of discontinuity the functions in these examples had.

9.4 Local Properties of Continuous Functions

The *local* properties of functions are those that are determined by the behavior of the function in an arbitrarily small neighborhood of the point in its domain of definition. For example, the continuity of a function at a point of its domain of definition is obviously a local property.

We shall now exhibit the main local properties of continuous functions.

[3] If a is a discontinuity, then a must be a limit point of the set D. It may happen, however, that all the points of D in some neighborhood of a lie on one side of a. In that case, only one of the limits in this definition is considered.

Theorem 24. Let $f\colon D \to \mathbb{R}$ be a function that is continuous at the point $a \in D$.Then the following statements hold.

1. The function $f\colon D \to \mathbb{R}$ is bounded in some neighborhood of a.
2. If $f(a) \neq 0$ then in some neighborhood of a all the values of the function have the same sign as $f(a)$.
3. If the real-valued function g is defined in some neighborhood of a and, like f, is continuous at a then the following function are defined in some neighborhood of a and continuous at a:
   (a) $(f+g)(x) = f(x) + g(x)$,
   (b) $(f \cdot g)(x) = f(x) \cdot g(x)$,
   (c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (provided $g(a) \neq 0$).
4. If the function $g\colon E \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at a point $b \in E$ and f is such that $f\colon D \to E$, $f(a) = b$, and f is continuous at a then the composite function $g \circ f$ is defined on E and continuous at a.

Proof. Parts 1, 2, and 3 follow from the corresponding properties of limits. Note that when proving Part 3c we need to explain that $\frac{f(x)}{g(x)}$ is defined in some neighborhood of a. But this *is* true by virtue of Part 2.

We now prove Part 4. By continuity of g at $b = f(a)$, for any fixed $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|g(f(x)) - g(f(a))| < \varepsilon. \tag{7}$$

as soon as

$$|f(x) - f(a)| < \delta \tag{8}$$

By continuity of f at a, for δ in (8) we can find $\delta_1 > 0$ such that $|f(x) - f(a)|$ as soon as $|x - a| < \delta_1$. Therefore $|x - a| < \delta_1$ implies (8) and hence (7). Since ε is arbitrary, it follows that $g(f(x)) \to g(f(a))$ as $x \to a$, i.e., $g \circ f$ is continuous at a. ■

9.5 Coninuity of Elementary Functions *Extra credit*

Functions you studied in your math classes before taking a Calculus course are called elementary functions. More precisely:

DEFINITION 39. A function is called an *elementary function* if it is one of the following kinds:

1. A rational function (ratio of two polynomials)

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{n-1} + \cdots + b_1 x + b_0},$$

   where at least one of the b_i, $0 \leqslant i \leqslant m$, is not zero (these include the *constant function* and the *polynomial function*);
2. An exponential function a^x, $a > 0$, $a \neq 1$;
3. A logarithmic function $\log_a x$, $a > 0$, $a \neq 1$;
4. Power and radical functions $\sqrt[n]{x}$ or x^b, $x > 0$. Note that these are compositions of the exponential function and the logarithmic function since $x^b = e^{b \ln x}$.
5. Trigonometric functions $\sin x$, $\cos x$, $\tan x$, etc;
6. The inverse of any of the above functions if it exists;
7. Any function obtained from the above functions as a finite number of sums, differences, products, quotients, or compositions.

Example 56. The function

$$f(x) = \frac{\arcsin\sqrt{1-x^2} + \tan x}{2^{-x} \cdot \ln x}$$

is an elementary function and as such it is continuous at every of its domain. What *is* the domain of f?

9.6 Problems

1.

10 Lecture 9 — Continuity (Part II)

Let f be a real-valued function defined in a neighborhood of a point $a \in \mathbb{R}$. In intuitive terms the function f is continuous at a if its value $f(x)$ approaches the value $f(a)$ that it assumes at the point a itself as x gets nearer to a.

10.1 Global Properties of Continuous Functions

A *global* property of a function, intuitively speaking, is a property involving the entire domain of definition of the function.

Theorem 25 (The Bolzano–Cauchy intermediate-value theorem)**.** If a function that is continuous on a closed interval assumes values with different signs at the endpoints of the interval, then there is a point in the interval where it assumes the value zero.

NOTE 28. When speaking about continuity of a function f on a closed interval $[a, b]$, $a < b$, we mean that f is continuous on (a, b), right-continuous at a, and left-continuous at b.

In logical symbols, this theorem has the following expression[4]

$$(f \in C[a, b] \wedge f(a) \cdot f(b) < 0) \Rightarrow \exists c \in [a, b]\, (f(c) = 0)\,.$$

To prove this theorem we will need what is known as the Nested Intervals Lemma (also known as Cauchy–Cantor Principle).

DEFINITION 40. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of sets. If $X_1 \supseteq X_2 \supseteq \cdots X_n \supseteq \cdots$, that is $X_n \supseteq X_{n+1}$ for all $n \in \mathbb{N}$, we say that the sequence is *nested*.

LEMMA 1 (Nested Intervals Lemma)**.** For any nested sequence $I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq \cdots$ of closed intervals, there exist a point $c \in \mathbb{R}$ belonging to all these intervals.

If in addition it is known that for any $\varepsilon > 0$ there is an interval I_k whose length $|I_k|$ is less than ε (that is, some subsequence of $|I_k|$ converges to 0 as $k \to \infty$) then c is the unique point common to all the intervals.

Proof. We first note that for any two closed intervals $I_m = [a_m, b_m]$ and $I_n = [a_n, b_n]$ of the sequence we have $a_m \leqslant b_n$ (that is, any left endpoint is to the left of any right endpoint). For if we suppose otherwise (that is, that $a_m > b_n$) then $a_n \leqslant b_n < a_m \leqslant b_m$ and so the intervals I_n and I_m are mutually disjoint, whereas one of them (the one with the larger index) is contained in the other.

We now apply the Axiom of Completeness of real numbers to the sets $A = \{a_m \mid m \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$ by virtue of which there is a number $c \in \mathbb{R}$ such that $a_m \leqslant c \leqslant b_n$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$. In particular, $a_n \leqslant c \leqslant b_n$ for all $n \in \mathbb{N}$. This clearly implies that $c \in I_n$ for all $n \in \mathbb{N}$.

Now if $c_1 < c_2$ are two points having this property, then $a_n \leqslant c_1 < c_2 \leqslant b_n$ for all $n \in \mathbb{N}$, and therefore $0 < c_2 - c_1 < b_n - a_n = |I_n|$, so that $|I_n| \geqslant c_2 - c_1$ for all $n \in \mathbb{N}$ violating the presence of intervals of infinitely small lengths in the sequence. ■

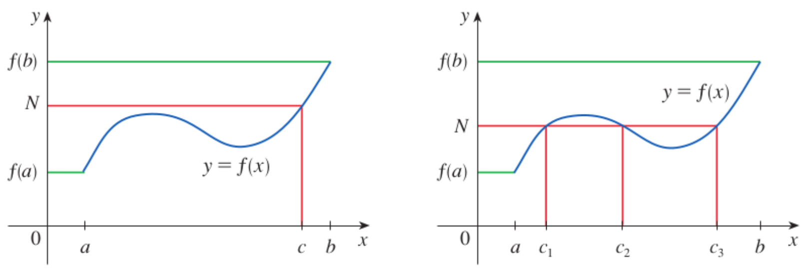
Proof of Theorem 25. Let us divide the interval $[a, b]$ in half. If the function does not assume the value 0 at the point of division then it must assume opposite values at the endpoints of one of the two subintervals. In that interval we proceed as we did with the original interval, that is, we bisect it and continue the process.

Then either at some step we hit a point $c \in [a, b]$ where $f(c) = 0$, or we obtain a sequence $\{I_n\}$ of nested closed intervals whose lengths tend to zero (why?!) and at whose endpoints f assumes values with opposite signs. In the second case, by the Nested Interval Lemma, there exists a unique point $c \in [a, b]$ common to all the intervals. By construction there are two sequences of endpoints $\{x'_n\}$ and $\{x''_n\}$ of the intervals I_n such that $f(x'_n) < 0$ and $f(x''_n) > 0$, while $\lim_{n\to\infty} x'_n = \lim_{n\to\infty} x''_n = c$. Then by continuity of f and properties of limit we have $\lim_{n\to\infty} f(x'_n) = f(c) \leqslant 0$ and $\lim_{n\to\infty} f(x''_n) = f(c) \geqslant 0$. Hence $f(c) = 0$. ■

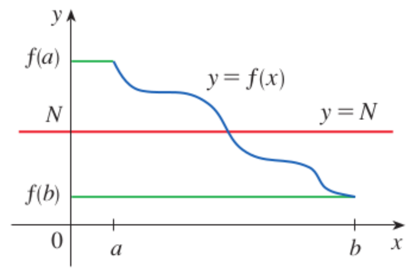
[4] We will write $C(D)$ to denote the set of all continuous functions on the set D. In the case $D = [a, b]$ we will write, more briefly, $C[a, b]$ instead of $C([a, b])$.

This is the most practically important form of the Intermediate-Value Theorem.

COROLLARY 4 (Intermediate-Value Theorem)**.** If the function f is continuous on an open interval and assumes values $f(a) = A$ and $f(b) = B$ at points a and b, then for any number N between A and B, there is a point c between a and b at which $f(c) = N$.



A continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. The value N can be taken on once or more than once.



If any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ then the graph of f can't jump over the line.

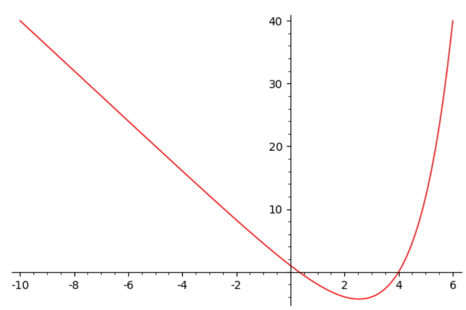
Proof of Corollaary 4. The function $f(x) - N$ is defined and continuous on the closed interval with endpoints a and b, and takes values of opposite signs at the endpoints. Applying Theorem 25, we conclude that there exists a point c between a and b such that $f(x) - N = 0$. ■

Here are some examples demonstrating the power of this theorem.

Example 57. The root $x = 4$ of the equation

$$2^x = 4x$$

is obvious but it is harder to spot another root. However setting $f(x) := 2^x - 4x$ we see that $f(0) = 1 > 0$ and $f(\frac{1}{2}) = \sqrt{2} - 2 < 0$, and so there exists a root in $(0, \frac{1}{2})$.



Example 58. Consider any polynomial equation $f(x) = 0$ where $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$ is a polynomial of *odd* power. Since, for sufficiently large positive x, $f(x)$ has the same sign as a_{2n+1}, and for sufficiently large negative x, the sign of $f(x)$ is opposite to that of a_{2n+1}. Thus there must exist a point at which f is zero.

Example 59. The Cauchy–Bolzano Theorem can be used to find roots of equation with arbitrary precision. Consider for instance the equation $f(x) := x^4 - x - 1 = 0$. As $f(1) = -1$ and $f(2) = 13$, there is a root of f in $(1, 2)$. Dividing the interval into 10 equal subintervals we find

$$f(1.1) = -0.63\ldots; \quad f(1.2) = -0.12\ldots; \quad f(1.3) = +0.55\ldots;$$

We now see that the root is in $(1.2, 1.3)$. Dividing this interval further into 10 subintervals we find that

$$f(1.21) = -0.06\ldots;\ f(1.22) = -0.004\ldots;\ f(1.23) = +0.058\ldots;$$

Now we know that the root is in $(1.22, 1.23)$. Continuing this way we can find an interval of arbitrarily small length containing the root.

Theorem 26 (Weierstrass Maximum-Value Theorem)**.** A function that is continuous on a closed interval is bounded on that interval. Moreover there is a point in the interval where the function assumes its maximum value and a point where it assumes its minimal value.

To prove this theorem we will need what is known as the Finite Covering Lemma (also known as Borel–Lebesgue Principle or Heine–Borel Lemma).

DEFINITION 41. A system $S = \{X\}$ of sets X is said to *cover* a set Y if $Y \subseteq \bigcup_{X\in S} X$, (that is, if every element $y \in Y$ belongs to at least one of the sets X in the system S). A subset of a set $S = \{X\}$ that is a system of sets will be called a *subsystem* of S.

LEMMA 2 (Borel–Lebesgue)**.** Every system of open intervals covering a closed interval contains a finite subsystem that covers the closed interval.

Proof. Let $S = \{U\}$ be a system of open intervals U that cover the closed interval $[a, b] = I_1$. If the interval I_1 could not be covered by a finite set of intervals of the system S then, dividing I_1 into two halves, we would find that at least one of the two halves, which we denote by I_2, does not admit a finite covering. We now repeat this procedure with the interval I_2, and so on.

In this way a nested sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of closed intervals arises, none of which admit a covering by a finite subsystem of S. Since the length of the interval I_n is $|I_n| = |I_1| \cdot 2^{-n}$, the sequence $\{I_n\}$ contains intervals f arbitrarily small length. But the Nested Interval Lemma implies that there exists a point c belonging to all of the intervals I_n, $n \in \mathbb{N}$. Sine $c \in I_1 = [a, b]$ there exists an open interval $(\alpha, \beta) = U \in S$ containing c, that is, $\alpha < c < \beta$. Let $\varepsilon = \min\{c - \alpha, \beta - c\}$. In the sequence just constructed, we find an interval I_n such that $|I_n| < \varepsilon$. Since $c \in I_n$ and $|I_n| < \varepsilon$, we conclude that $I_n \subseteq U = (\alpha, \beta)$. But this contradicts the fact that the interval I_n cannot be covered by a finite set of intervals from the system. ∎

NOTE 29. Both the condition that the interval be closed and the cover be open intervals are important. For instance the system of open intervals

$$\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{8}, \frac{3}{8}\right), \ldots, \left(\frac{1}{2^n}, \frac{3}{2^n}\right), \ldots$$

cover the the set $(0, 1]$, but no finite subsystem does. Likewise, the system

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \ldots, \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}\right], \ldots \text{ and } [1, 2]$$

cover $[0, 2]$, but no finite subsystem does.

Proof of Theorem 26. Let $f : D \to \mathbb{R}$ be a continuous function on the close interval $D = [a, b]$. By the local properties of a continuous function for any point $x \in D$ there exists a neighborhood $U(x)$ such that the function is bounded on the set $D \cap U(x)$. The set of such neighborhoods $U(x)$ constructed for all $x \in D$ forms a covering of the closed interval $[a, b]$ by open intervals. By the Finite Covering Lemma, one can extract a finite system $U(x_1), \ldots, U(x_n)$ of open itnervals that together form cover the closed interval $[a, b]$. Since the function is bounded on each set $D \cap U(x_k)$, that is, $m_k \leqslant f(x) \leqslant M_k$, where m_k and M_k are real numbers and $x \in U(x_k)$, we have

$$\min\{m_1, \ldots, m_k\} \leqslant f(x) \leqslant \max\{M_1, \ldots, M_k\}$$

for *all* $x \in D$. It is now established that $f(x)$ is bounded on D.

Now let $M = \sup_{x\in D} f(x)$. Assume that $f(x) < M$ for all $x \in D$. Then the continuous function $M - f(x)$ on D is nowhere zero, although (by the definition of M) it assumes values arbitrarily close to zero. It the follows that the function $\frac{1}{M - f(x)}$ is, on the one hand, continuous on D because of the local properties of continuous functions, but on the other hand not bounded on D, which contradicts what has just been proved about a function continuous on a closed interval.

Thus there must be a point $x_M \in [a, b]$ at which $f(x_M) = M$.

Similarly, by considering $m = \inf_D f(x)$ and the auxiliary function $\frac{1}{f(x) - m}$, we prove that there exists a point $x_m \in [a, b]$ at which $f(x_m) = m$. ∎

NOTE 30. The functions $f_1(x) = x$ and $f_2(x) = \frac{1}{x}$ are continuous on the open interval $D = (0, 1)$ but f_1 has neither a maximal nor a minimal value on D, and f_2 is unbounded on D. Thus, the properties of a continuous function expressed in Theorem 26 involve some property of the domain of definition, namely the property that from every covering of D by open intervals one can extract a finite subcovering. From now on we shall call such sets *compact*.

Theorem 27. A (nonconstant) continuous function maps a closed interval into a closed interval.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a nonconstant continuous function and let $E := \{f(x) \mid x \in [a, b]\}$ be the set of values taken by f. Since f is continuous, $E \subseteq \mathbb{R}$ is a bounded set, that is, both $M := \max E$ and $m := \min E$ are in $\mathbb{R}$. Moreover, $m < M$ and $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$, i.e. $E \subseteq [m, M]$. Conversely, we note that both m and M are values of f and so is every number between them. That is, $[m, M] \subseteq E$. Hence $E = [m, M]$. ■

10.2 Problems

1.

11 Lecture 10 — The Derivative Concept

TO DO: Introduce the differential notation as $\mathrm{d}f_x(h)$ rather than $\mathrm{d}f(x)(h)$.

Let f be a real-valued function defined in a neighborhood of a point $a \in \mathbb{R}$. In intuitive terms the function f is continuous at a if its value $f(x)$ approaches the value $f(a)$ that it assumes at the point a itself as x gets nearer to a.

11.1 Definition and Notation

DEFINITION 42. A function $f\colon D \to \mathbb{R}$ defined on a set $D \subseteq \mathbb{R}$ is *differentiable* at a point $x \in D$ that is a limit point of D if

$$f(x+h) - f(x) = A(x)h + \phi(x;h)h, \tag{9}$$

where $h \mapsto A(x)h$ is a linear function in h and $\phi(x;h) \to 0$ as $h \to 0$, $x + h \in D$.

The quantities

$$\Delta x(h) := (x+h) - x = h$$

and

$$\Delta f(x;h) := f(x+h) - f(x)$$

are called respectively the *increment of the argument* and the *increment of the function* (corressponding to this increment in the argument).

The function $h \mapsto A(x)h$ is called the *differential of the function* $f\colon D \to \mathbb{R}$ at the point $x \in D$ and is denoted $\mathrm{d}f(x)$, i.e. $\mathrm{d}f(x)(h) = A(x)h$.

From (9) it follows that

$$0 = \lim_{h\to 0} \phi(x;h) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} - A(x)\right)$$

and so

$$A(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

The number A is called the *derivative* of the function f at the point x denoted $f'(x)$.

It follows immediately that

Theorem 28. A function that is differentiable at a point is continuous at that point.

NOTE 31. The converse, that is, the statement that a function that is continuous at a point is differentiable at that point, is false. Think of $|x|$.

NOTE 32. Various relations between the concepts just defined are easy to derive. For instance

$$\Delta f(x;h) - \mathrm{d}f(x)(h) = f(x+h) - f(x) - \mathrm{d}f(x)(h) = \phi(x;h)\cdot h,$$

that is, the difference between the increment of the function due to the increment h in its argument and the value of the function $\mathrm{d}f(x)$, which is linear in h, at the same h, is an infinitesimal of "higher order than the first" in h. Also

$$\mathrm{d}f(x)(h) \overset{\text{def}}{=} A(x)h = f'(x)h.$$

In particular, if $f(x) \equiv x$, we have $f'(x) = 1$ and

$$\mathrm{d}x(x)(h) = h,$$

so that it is sometimes said that "the differential of an independent variable equals its increment". Using we may write

$$\mathrm{d}f(x)(h) = f'(x)h = f'(x)\,\mathrm{d}x(h).$$

Note that this is true for any h such that $x + h \in D$, and so we have the equality of functions

$$\mathrm{d}f(x) = f'(x)\,\mathrm{d}x.$$

It follows also that

$$f'(x) = \frac{\mathrm{d}f(x)(h)}{\mathrm{d}x(h)},$$

that is, the function $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ (the ratio of the functions $\mathrm{d}f(x)$ and $\mathrm{d}x$) is constant and equals $f'(x)$. For this reason, following Leibnitz, we frequently denote the derivative by the symbol $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$, alongside the notation $f'(x)$ proposed by Lagrange.

11.2 The Tangent Line; Geometric Meaning of the Derivative and Differential

Let $f : D \to \mathbb{R}$ be a function defined on a set $D \subseteq \mathbb{R}$ and x_0 is a limit point of D. We wish to choose the constant c_0 so as to give the best possible description of the behavior of the function in a neighborhood of the point x_0 among constant functions. More precisely, we want the difference $f(x) - c_0$ to be infinitesimal compared with any nonzero constant as $x \to x_0$, $x \in D$, that is

$$f(x) = c_0 + \alpha(x; x_0),$$

where $\alpha(x; x_0) \to 0$ as $x \to x_0$. This last relation is equivalent to saying that $\lim_{x\to x_0} f(x) = c_0$. If in particular f is continuous at x_0 then $\lim_{x\to x_0} f(x) = f(x_0)$ and naturally $c_0 = f(x_0)$.

Now let us try to choose a *linear* function $x \mapsto c_0 + c_1(x - x_0)$ so as to have

$$f(x) = c_0 + c_1(x - x_0) + \beta(x; x_0)(x - x_0),$$

where $\beta(x; x_0) \to 0$ as $x \to x_0$. It follows immediately that $c_0 = \lim_{x\to x_0} f(x)$ and if the function is continuous at this point then $c_0 = f(x_0)$. In this case (writing $h = x - x_0$ and noting that $x \to x_0$ is equivalent to $h \to 0$) we have

$$f(x_0 + h) - f(x_0) = c_1 h + \beta(x; h)h,$$

which is equivalent to the condition that $f(x)$ is differentiable at x_0. From this we find

$$c_1 = \frac{f(x_0 + h) - f(x_0)}{h} - \beta(x; h) \Longrightarrow c_1 = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

We have thus proved the following

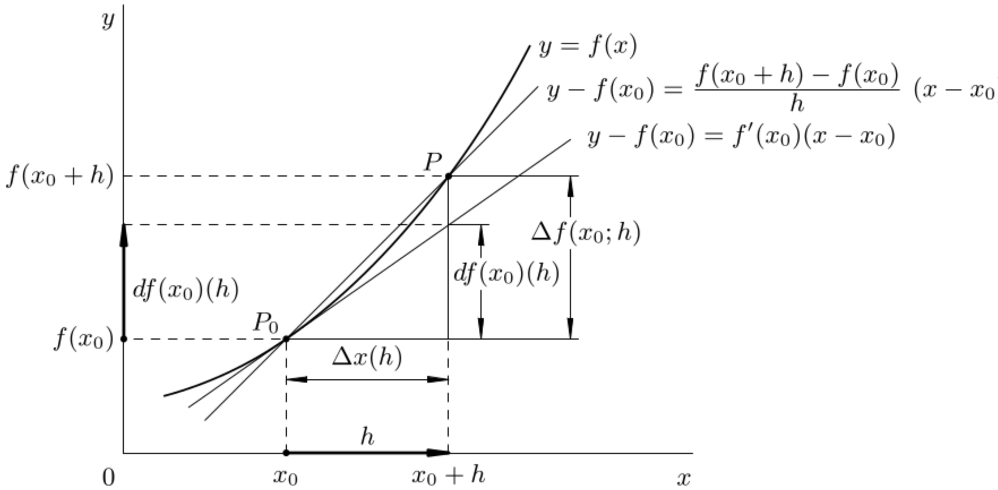
Proposition 1. A function $f : D \to \mathbb{R}$ that is continuous at a point $x_0 \in D$ that is a limit of $D \subseteq \mathbb{R}$ admits a linear approximation if and only if it is differentiable at the point.

This linear approximation is provided by the function

$$\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$$

whose graph is the straight line passing through the point $(x_0, f(x_0))$ and having slope $f'(x_0)$.

DEFINITION 43. If a function $f : D \to \mathbb{R}$ is defined on a set $D \subseteq \mathbb{R}$ and differentiable at a point $x_0 \in D$, the line defined by the equation $y - f(x_0) = f'(x_0)(x - x_0)$ is called the *tangent line* to the graph of this function at the point $(x_0, f(x_0))$.



The figure illustrates all the basic concepts we have so far introduced in connection with differentiability of a function at a point: the increment of the argument, the increment of the function corresponding to it, and the value of the differential. The figure shows the graph of the function, the tangent to the graph at the point $P_0 = (x_0, f(x_0))$, and for comparison, an arbitrary line (usually called a *secant*) passing through P_0 and some point $P \neq P_0$ of the graph of the function.

11.3 Problems

1.

12 Lecture 11 — Derivatives of Some Elementary Functions

1. *The constant function:* $\boxed{(c)' = 0}$.

Proof. Setting $f(x) = c$,

$$f'(c) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

■

2. *The power function:* $\boxed{(x^\mu)' = \mu x^{\mu-1}}$, where the domain of the power function depends on μ. For example, if μ is a positive integer then $x \in \mathbb{R}$; if μ is a negative integer then $x \in \mathbb{R} \setminus \{0\}$; if $\mu = \frac{1}{m}$, where m is a positive integer then $x \in \mathbb{R}$ if m is odd and $x \in [0, +\infty)$ if m is even. In particular it follows that

$$\boxed{\left(\frac{1}{x}\right)' = -\frac{1}{x^2}}, \quad \boxed{\left(\sqrt{x}\right)' = -\frac{1}{2\sqrt{x}}}.$$

Proof. Setting $f(x) = x^\mu$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^\mu - x^\mu}{h} = x^\mu \cdot \lim_{h\to 0} \frac{\frac{(x+h)^\mu}{x^\mu} - 1}{h} = x^{\mu-1} \cdot \lim_{h\to 0} \frac{\left(1 + \frac{h}{x}\right)^\mu - 1}{\frac{h}{x}}. \qquad (10)$$

We shall now prove that $\lim_{t\to 0} \frac{(1+t)^\mu - 1}{t} = \mu$ for any $\mu \in \mathbb{R}$ (Note that we already dealt with a particular instance $\mu \in \mathbb{Q}$ of this limit in the past.) First of all note that, by continuity of logarithm as an elementary function,

$$\lim_{t\to 0} \frac{\log_a(1+t)}{t} = \lim_{t\to 0} \log_a (1+t)^{\frac{1}{t}} = \log_a \lim_{t\to 0} (1+t)^{\frac{1}{t}} = \log_a \mathrm{e}.$$

In particular, $\lim_{t\to 0} \frac{\ln(1+t)}{t} = 1$. Now letting $s := (1+t)^\mu - 1$ we note that $s \to 0$ as $t \to 0$ by continuity of the power function. Taking natural logs of both sides of the equality $s + 1 = (1+t)^\mu$ we find

$$\ln(s+1) = \mu \ln(1+t).$$

Now we have

$$\frac{(1+t)^\mu - 1}{t} = \frac{s}{t} = \frac{s}{\ln(s+1)} \cdot \frac{\ln(s+1)}{\ln(t+1)} \cdot \frac{\ln(t+1)}{t} = \frac{s}{\ln(s+1)} \cdot \mu \cdot \frac{\ln(t+1)}{t} \xrightarrow[t\to 0]{} 1 \cdot \mu \cdot 1 = \mu.$$

Now continuing (10) we have

$$f'(x) = x^{\mu-1} \cdot \lim_{h\to 0} \frac{\left(1 + \frac{h}{x}\right)^\mu - 1}{\frac{h}{x}} = \mu x^{\mu-1}.$$

■

3. *The exponential function:* $\boxed{(a^x)' = a^x \ln a}$. In particular, it follows that $\boxed{(\mathrm{e}^x)' = \mathrm{e}^x}$.

Proof. Setting $f(x) = a^x$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}. \tag{11}$$

We now show that $\lim_{t\to 0} \frac{a^t-1}{t} = \ln t$. Taking $b := a^t - 1$ we see that, by continuity of the exponential function, $t \to 0$ implies $b \to 0$. As $t = \log_a(1+b)$, we have

$$\lim_{t\to 0} \frac{a^t - 1}{t} = \lim_{b\to 0} \frac{b}{\log_a(1+b)} = \frac{1}{\log_a \mathrm{e}} = \ln a.$$

Now continuing (11) we have

$$f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \ln a.$$

■

4. *The Logarithmic Function:* **Change to** $\ln|x|$. $\boxed{(\log_a x)' = \frac{\log_a \mathrm{e}}{x}}$. In particular, $\boxed{(\ln x)' = \frac{1}{x}}$.

Proof. Setting $f(x) = \log_a x$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h} = \frac{1}{x} \cdot \lim_{h\to 0} \frac{\log_a\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} = \frac{\log_a \mathrm{e}}{x}. \tag{12}$$

■

5. *Trigonometric functions:*

$$\boxed{(\sin x)' = \cos x}, \quad \boxed{(\cos x)' = -\sin x}, \quad \boxed{(\tan x)' = \frac{1}{\cos^2 x}}.$$

Proof. Letting $f(x) = \sin x$ we show that $f'(x) = \cos x$. Using the formula $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$, we find

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x + \frac{h}{2}\right)}{h} \\ &= \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos x. \end{aligned}$$

For $f(x) = \tan x$ we would have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} = \\ &= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)} = \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos x\cos(x+h)} = \frac{1}{\cos^2 x}. \end{aligned}$$

■

We also give examples of computing derivatives of non-elementary functions.

1. Let $f(x) = |x|$. Then at the point $x_0 = 0$ we have

$$\lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0-} \frac{|x| - 0}{x - 0} = \lim_{x\to x_0-} \frac{-x}{x} = -1,$$

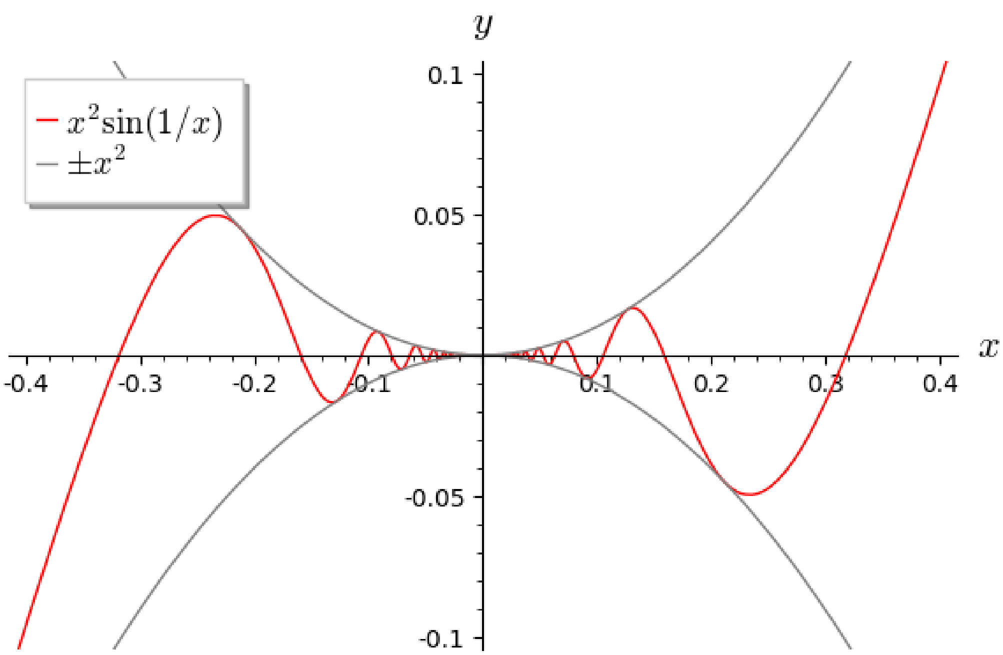
$$\lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0+} \frac{|x| - 0}{x - 0} = \lim_{x\to x_0+} \frac{x}{x} = 1.$$

Consequently, at this point the function has no derivative and hence is not differentiable at the point.

2. Let the function be given by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Here is the graph of this function.



Let us find the tangent to the graph at the point $(0, 0)$. Since

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} x \sin\frac{1}{x} = 0,$$

the tangent has the equation $y - 0 = 0 \cdot (x - 0)$, or simply $y = 0$. Thus, in this example the tangent is the x-axis, which the graph intersects infinitely many times in any neighborhood of the point of tangency.

12.1 Problems

1. Use the definition of derivative and relevant trigonometric formulas to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$.

13 Lecture 12 — Rules of Differentiation

13.1 Differentiation and the Arithmetic Operations

Theorem 29. If functions $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are differentiable at a point $x \in D$ then

1. their sum is differentiable at x, and
$$(f+g)'(x) = (f'+g')(x);$$
2. their product is differentiable at x, and
$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x);$$
3. their quotient is differentiable at x, and
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. In the proof we rely of course on the definition of derivative of the function f at a point x_0:

$$f'(x) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x) - f(x_0)}{x - x_0}.$$

For part 1 we have

$$\begin{aligned}
(f+g)'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \left\{ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right\} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

For part 2 we have

$$\begin{aligned}
(f\cdot g)'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) + f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \left\{ f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right\} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

For part 3 we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\left(\frac{f}{g}\right)(x+h)-\left(\frac{f}{g}\right)(x)}{h}\\ &= \lim_{h\to 0}\frac{\left(\frac{f}{g}\right)(x+h)-\left(\frac{f}{g}\right)(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}\\ &= \lim_{h\to 0}\left\{\frac{g(x)(f(x+h)-f(x))}{hg(x+h)g(x)}-\frac{f(x)(g(x+h)-g(x))}{hg(x+h)g(x)}\right\}\\ &= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.\end{aligned}$$

■

DEFINITION 44. Given functions $f_1(x), f_2(x), \dots, f_n(x)$, $n\in\mathbb{N}$, any expression of the form $c_1f_1(x)+c_2f_2(x)\cdots c_nf_n(x)$ is called a *linear combination* of the f_i, $1\leqslant i\leqslant n$.

COROLLARY 5. The derivative of a linear combination of differentiable functions equals the same linear combination of derivatives of these functions. More precisely, if f_i, $1\leqslant i\leqslant n$, $n\in\mathbb{N}$, are functions differentiable at a point x which is a limit of the common domain of the f_i and $c_i\in\mathbb{R}$, $1\leqslant i\leqslant n$, are arbitrary then

$$\left(\sum_{i=1}^{n}c_if_i(x)\right)' = \sum_{i=1}^{n}c_if_i'(x).$$

Proof. Note first of all that by Theorem 2, we have for any constant $c\in\mathbb{R}$ and any function f differentiable at a point x which is a limit of $\operatorname{Dom}(f)$

$$(cf(x))' = (c)'f(x)+c(f(x))' = 0\cdot f(x)+c\cdot f'(x) = cf'(x).$$

Now for $n=2$ we would have

$$(c_1f_1(x)+c_2f_2(x))' = (c_1f_1(x))'+(c_2f_2(x))' = c_1f_1'(x)+c_2f_2'(x).$$

The result now follows by the method of mathematical induction. ■

COROLLARY 6. If the functions $f_1, \dots, f_n$ are differentiable at x then

$$(f_1\cdots f_n)'(x) = f_1'(x)f_2(x)\cdots f_n(x)+f_1(x)f_2'(x)f_3(x)\cdots f_n(x)+\cdots+f_1(x)\cdots f_{n-1}(x)f_n'(x)$$

COROLLARY 7. It follows from the relation between the derivative and the differential that Theorem 29 can also be written in terms of differentials. To be specific:

1. $\mathrm{d}(f+g)(x) = \mathrm{d}f(x)+\mathrm{d}g(x)$;
2. $\mathrm{d}(f\cdot g)(x) = g(x)\mathrm{d}f(x)+f(x)\mathrm{d}g(x)$;
3. $\mathrm{d}\left(\frac{f}{g}\right)(x) = \dfrac{g(x)\mathrm{d}f(x)-f(x)\mathrm{d}g(x)}{g^2(x)}$.

Proof. We only verify statement 1, and the proof of the other two statements is similar.

$$\mathrm{d}(f+g)(x)h = (f+g)'(x)h = (f'+g')(x)h = (f'(x)+g'(x))h = f'(x)h+g'(x)h = \mathrm{d}f(x)h+\mathrm{d}g(x)h,$$

and we have verified that $\mathrm{d}(f+g)(x)$ and $\mathrm{d}f(x)+\mathrm{d}g(x)$ are the same function. ■

13.2 Examples

1. Differentiate $f(x) = 3x^4 - \sqrt{x}$.
2. Differentiate $f(x) = x^2 \sin ax$.
3. Differentiate $f(x) = \sqrt{x} \sin x \cos x$.
4. Differentiate $f(x) = \cot x$.
5. Differentiate the polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where all $a_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$.
6. Differentiate the rational function $f(x) = \frac{x^2 - 5x}{x^3 + 3}$.

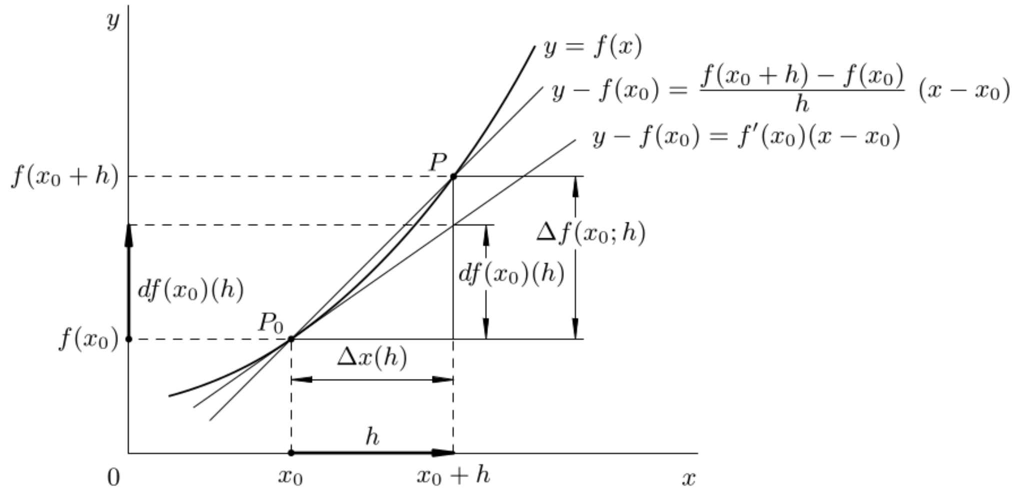
13.3 Problems

1. Use the definition of derivative and relevant trigonometric formulas to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$.

14 Lecture 13 — Differentiation of a Composite Function

14.1 Differentiation of a Composite Function (Chain Rule)

Recall first of all the beautiful picture from before:



NOTE 33. The number $h = x - x_0$, that is, the increment of the argument, can be regarded as a vector attached to the point x_0 and defining the transition from x_0 to $x = x_0 + h$. We denote the set of all such vectors by $T\mathbb{R}(x_0)$. Similarly, we denote by $T\mathbb{R}(y_0)$ the set of all displacement vectors from the point y_0 along the y-axis. It can then be seen from the definition of the differential that the mapping

$$\mathrm{d}f(x_0)\colon T\mathbb{R}(x_0) \to T\mathbb{R}f(x_0).$$

We remark that if the mapping

$$h \mapsto f(x_0 + h) - f(x_0) = \Delta f(x_0; h)$$

is the increment of the ordinate of the graph of the function $y = f(x)$ as the argument passes from x_0 to $x_0 + h$, then the differential gives the increment in the ordinate of the tangent to the graph of the function for the same increment h in the argument.

Theorem 30 (The Chain Rule)**.** If the function $f\colon X \to Y \subset \mathbb{R}$ is differentiable at a point $x \in X$ and the function $g\colon Y \to \mathbb{R}$ is differentiable at the point $y = f(x) \in Y$, then the composite function $g \circ f\colon X \to \mathbb{R}$ is differentiable at x, and the differential $\mathrm{d}(g \circ f)\colon T\mathbb{R}(x) \to T\mathbb{R}(g(f(x)))$ of their composition equals the composition $\mathrm{d}g(y) \circ \mathrm{d}f(x)$ of theire differential

$$\mathrm{d}f(x)\colon T\mathbb{R}(x) \to T\mathbb{R}(y = f(x)) \quad \text{and} \quad \mathrm{d}g(y = f(x))\colon T\mathbb{R}(y) \to T\mathbb{R}(g(y)).$$

Proof. The condition for differentiability of the functions f and g at respectively x and y have the form

$$f(x + h) - f(x) = f'(x)h + \alpha(h) \cdot h,$$
$$g(y + t) - g(y) = g'(y)t + \beta(t) \cdot t,$$

where $x + h \in X$, $y + t \in Y$, $\alpha(h) \to 0$ and $\beta(h) \to 0$ as $h \to 0$. (Note that previously in the definition of differentiability we used to emphasize the $\alpha = \alpha(x; h)$ and $\beta = \beta(y; t)$ depend also on the points x and y. Here however for brevity of presentation we omit this.)

First of all note that the second equation above holds for any choice of $\beta(0)$ so we as well assume that $\beta(0) = 0$ so β is continuous at $h = 0$. Now with $y = f(x)$ and setting $f(x+h) = y + t$ we see that $t \to 0$ as $h \to 0$ as f is continuous at x. Define

$$\gamma(h) \overset{\text{def}}{=} \beta(f(x+h) - f(x)), \quad \text{so} \quad \lim_{h\to 0} \gamma(h) = 0.$$

$$\begin{aligned}
(g \circ f)(x+h) - (g \circ f)(x) &= g\Big(f(x+h)\Big) - g\Big(f(x)\Big) \\
&= g(y+t) - g(y) \\
&= g'(y)t + \beta(t) \cdot t \\
&= g'\Big(f(x)\Big) \cdot \Big(f(x+h) - f(x)\Big) + \beta\Big(f(x+h) - f(x)\Big) \cdot \Big(f(x+h) - f(x)\Big) \\
&= g'\Big(f(x)\Big) \cdot \Big(f'(x)h + \alpha(h) \cdot h\Big) + \gamma(h)\Big(f'(x) + \alpha(h)\Big) \cdot h \\
&= g'\Big(f(x)\Big) \cdot f'(x) \cdot h + \underbrace{\left(g'\Big(f(x)\Big) \cdot \alpha(h) + \gamma(h)\Big(f'(x) + \alpha(h)\Big)\right)}_{\delta = \delta(h) \text{ is infinitesimal at } 0 \text{ as a function of } h} \cdot h,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\beta(t) &= \beta\Big(f(x+h) - f(x)\Big) \cdot \Big(f(x+h) - f(x)\Big) \\
&= \gamma(h) \cdot \Big(f'(x)h + \alpha(h) \cdot h\Big) \\
&= \gamma(h) \cdot \Big(f'(x) + \alpha(h)\Big) \cdot h.
\end{aligned}$$

Note now that

$$\left(\mathrm{d}g\Big(f(x)\Big) \circ \mathrm{d}f(x)\right)(h) = \mathrm{d}g\Big(f(x)\Big)\left(\mathrm{d}f(x)(h)\right) = \mathrm{d}g(f(x))\left(f'(x) \cdot h\right) = g'(f(x)) \cdot f'(x) \cdot h$$

is a composition of maps

$$h \overset{\mathrm{d}f(x)}{\longmapsto} f'(x)h, \quad \tau \overset{\mathrm{d}g(y)}{\longmapsto} g'(y)\tau$$

so that

$$h \overset{\mathrm{d}g(y)\circ \mathrm{d}f(x)}{\longmapsto} f'(g(x))g'(x)h.$$

We have thus proved that

$$(g \circ f)(x+h) - (g \circ f)(x) = g'(f(x)) \cdot f'(x) \cdot h + \delta(h) \cdot h,$$

where $\delta(h) \to 0$ as $h \to 0$. ■

This proof is due A. Kasiukov. In this proof, instead of $\mathrm{d}f(x)(h)$ we shall write $\mathrm{d}_x f(h)$. *Consider rewriting these notes using this notation!*

Proof. By the definition of differentiability of f at x may be written as

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - \mathrm{d}_x f(h)}{h} = 0.$$

We now have

$$\lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))-(\mathrm{d}_{f(x)}g\circ \mathrm{d}_x f)(h)}{h}\right| = \lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))-\mathrm{d}_{f(x)}g\Big(\mathrm{d}_x f(h)\Big)}{h}\right| =$$

$$= \lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))-\mathrm{d}_{f(x)}g\Big(f(x+h)-f(x)\Big)+\mathrm{d}_{f(x)}g\Big(f(x+h)-f(x)\Big)-\mathrm{d}_{f(x)}g\Big(\mathrm{d}_x f(h)\Big)}{h}\right| \leqslant$$

$$\leqslant \lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))-\mathrm{d}_{f(x)}g\Big(f(x+h)-f(x)\Big)}{h}\right| + \lim_{h\to 0}\left|\frac{\mathrm{d}_{f(x)}g\Big(f(x+h)-f(x)\Big)-\mathrm{d}_{f(x)}g\Big(\mathrm{d}_x f(h)\Big)}{h}\right| \leqslant$$

$$\leqslant \lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))-\mathrm{d}_{f(x)}g\Big(f(x+h)-f(x)\Big)}{h}\right| + \left|\mathrm{d}_{f(x)}g\left(\lim_{h\to 0}\left|\frac{f(x+h)-f(x)-\mathrm{d}_x f(h)}{h}\right|\right)\right| =$$

$$= 0 + |\mathrm{d}_{f(x)g}(0) = 0,$$

where the last equality is due the linearity of the differential $\mathrm{d}_{f(x)}g$, the f being differentiable at x, and the function g being differentiable at $f(x)$.

It now follows that

$$\mathrm{d}_x(g\circ f) = \mathrm{d}_{f(x)}g\circ \mathrm{d}_x f,$$

and the proof is over. ■

COROLLARY 8. The derivative $(g\circ f)'(x)$ of the composition of differentiable real-valued functions equals the product $g'(f(x))\cdot f'(x)$ of the derivatives of these functions computed at the corresponding points.

NOTE 34. There is a strong temptation to give a short proof of this last statement in Leibniz' notation for the derivative, in which if $z = z(y)$ and $y = y(x)$, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x},$$

which appears to be completely natural, if one regards the symbol $\frac{\mathrm{d}z}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ not as a unit, but as the ratio of $\mathrm{d}z$ to $\mathrm{d}y$ to $\mathrm{d}x$.

The idea for a proof that thereby arises is to consider the difference quotient

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\cdot\frac{\Delta y}{\Delta x}$$

and then pass to the limit as $\Delta x\to 0$. The difficulty that arises here is that Δy may be 0 even if $\Delta x\neq 0$.

Problem 2 (Very good!)**.** Go over the proof of the counterpart of this theorem in Silverman (Theorem 5.6).

COROLLARY 9. If the composition $(f_n\circ\cdots f_1)(x)$ of differentiable functions $y_1 = f_1(x)$, $y_2 = f_2(y_1)$,..., $y_n = f_n(y_{n-1})$ exists then

$$(f_n\circ\cdots f_1)'(x) = f_n'(y_{n-1})f_{n-1}'(y_{n-2})\cdots f_2'(y_1)f_1'(x).$$

14.2 Examples

1. Differentiate $f(x) = \sin(x^2)$.
2. Differentiate $f(x) = \sqrt{1 - x^2}$.
3. Differentiate $f(x) = (x^2 + x + 1)^n$.
4. Differentiate $f(x) = 2^{\sin x}$.
5. **Replace by** $\mathrm{d}\left(\ln|f(x)|\right)(x)$. Compute $\mathrm{d}\left(\ln f(x)\right)(x)$ for any function f differentiable in at x. This result plays a major role in Calculus II.
6. Differentiate a function $f(x) = u(x)^{v(x)}$, where $u(x)$ and $v(x)$ are differentiable functions and $u(x) > 0$.
7. Differentiate $f(x) = \sqrt{\tan \frac{1}{2} x}$.
8. Differentiate $f(x) = \mathrm{e}^{\sin^2 \frac{1}{x}}$.
9. Differentiate $f(x) = \mathrm{e}^{\sin^2 \frac{1}{x}}$.
10. Differentiate $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.
11. Differentiate $f(x) = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{x}}}$.
12. Differentiate for $a, b > 0$ the function $f(x) = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b$.
13. Differentiate for $a > 0$ the function $f(x) = x^{a^a} + a^{x^a} + a^{a^x}$.
14. Differentiate $f(x) = x + x^x + x^{x^x}$.
15. Differentiate for $a > 0$ the function $f(x) = x^{x^a} + x^{a^x} + a^{x^x}$.

14.3 Problems

1.

15 Lecture 14 — Differentiation of an Inverse Function

15.1 About Inverse Functions

DEFINITION 45 (Surjective Map)**.** A mapping $f\colon X \to Y$ is said to be *surjective* if $f(X) = Y$. Put another way: a mapping is surjective if

$$\forall y \in Y\ \exists x \in X\ :\ f(x) = y.$$

DEFINITION 46 (Injective Map)**.** A mapping $f\colon X \to Y$ is said to be *injective* if for any elements x_1, x_2 of X

$$\Big(f(x_1) = f(x_2)\Big) \Rightarrow (x_1 = x_2).$$

Put another way: a mapping is injective if

$$\forall x_1, x_2 \in X\ (x_1 \neq x_2) \Rightarrow \Big(f(x_1) \neq f(x_2)\Big).$$

That is, distinct elements have distinct images.

DEFINITION 47 (Bijective Map)**.** A mapping is called *bijective* if it is both injective and surjective.

NOTE 35. It should be emphasized that whether a function $f\colon X \to Y$ is bijective depends not only on f itself, but also on X and Y. Consider, for instance, $f(x)$ for which $x \mapsto x^2$, and

(i) $X = [0, \infty)$, $Y = \mathbb{R}$.

(ii) $X = \mathbb{R}$, $Y = [0, \infty)$.

(iii) $X = Y = [0, \infty)$.

If the mapping $f\colon X \to Y$ is bijective, that is, it is a one-to-one correspondence between the elements of the sets X and Y, there naturally arises a mapping

$$f^{-1}\colon Y \to X,$$

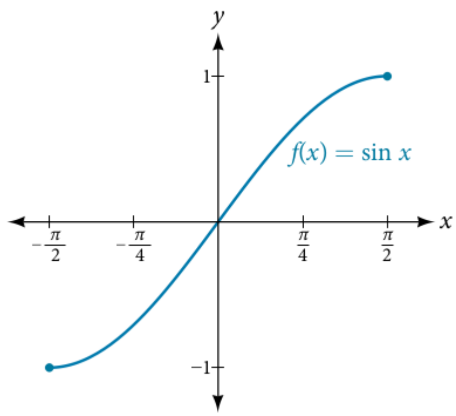
defined as follows: if $f(x) = y$, then $f^{-1}(y) = x$, that is, to each element $y \in Y$ one assigns the element $x \in X$ whose image under the mapping f is y. By the surjectivity of f there exists such an element, and by the injectivity of f, it is unique. Hence the mapping f^{-1} is *well-defined* (Do give an example of a mapping that is *not* well-defined.) This mapping is called the *inverse* of the original mapping f.

It is clear from the construction of the inverse mapping that $f^{-1}\colon Y \to X$ is itself bijective and that its inverse $(f^{-1})^{-1}\colon X \to Y$ is the same as the original mapping $f\colon X \to Y$. Thus the property of two mappings of being inverses is reciprocal: if f^{-1} is inverse for f, then f is inverse for f^{-1}.

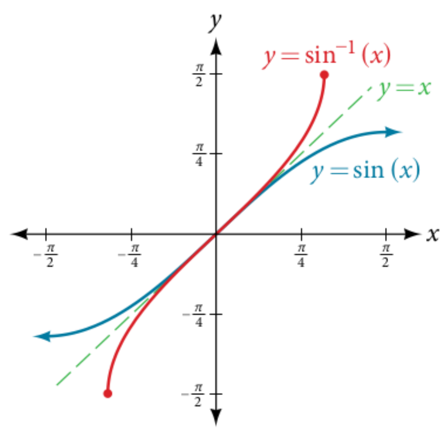
Problem 3. Suppose that the function $f\colon X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}$ has the inverse function f^{-1}. Show that the graphs of f and f^{-1}, when drawn in the standard Cartesian plane, are symmetric about the line $y = x$. (Hint: Let (a, b) be a point on the graph of f so $b = f(a)$. Then $a = f^{-1}(b)$, and so (b, a) is a point on the graph of f^{-1}. Consider the line through the point (a, b) and (b, a): 1) show that this line is perpendicular to the line $y = x$; 2) show that the mindpoint of the line segment joining the points (a, b) and (b, a) lies on the line $y = x$. These two facts combined imply that the graphs of f and f^{-1} are symmetric about the line $y = x$.)

Example 60. The most important examples of inverse functions in Calculus are inverse trigonometric functions.

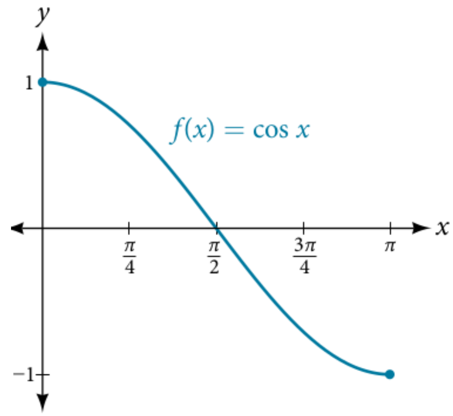
- The *inverse sine function* $f(x) = \arcsin x$: we restrict the function $\sin x$, which is *not* bijective (not injective) when defined on all of $\mathbb{R}$, to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, obtaining thereby a bijective function from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$:



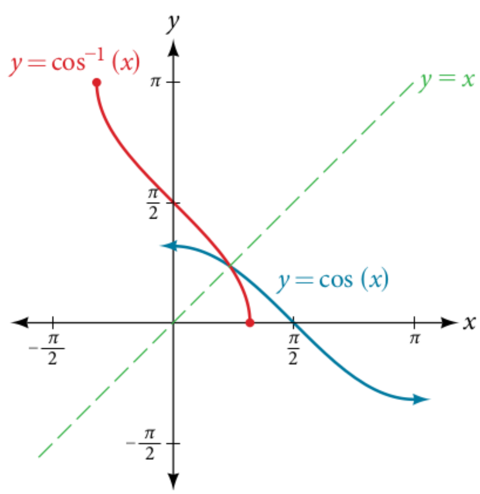
Thus given $y \in [-1, 1]$, the unique solution $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ of the equation $\sin x = y$ is called *arcsine* or *inverse sine* of y, denoted $\arcsin y$ or $\sin^{-1} y$. This mapping $y \mapsto x$ with $\sin x = y$ has domain $[-1, 1]$ and image $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The graph of the arcsine function when drawn in the usual Cartesian plane is symmetric to the graph of $\sin x$ with respect to the line $y = x$.



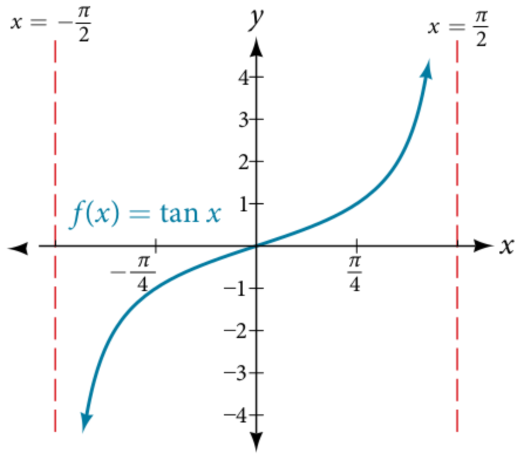
- The *inverse cosine function* $f(x) = \arccos x$: we restrict the function $\cos x$, which is *not* bijective (not injective) when defined on all of $\mathbb{R}$, to the interval $[0, \pi]$, obtaining thereby a bijective function from $[0, \pi]$ to $[-1, 1]$:



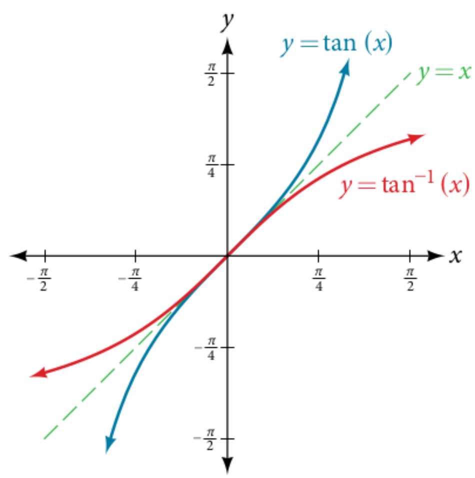
Thus given $y \in [-1, 1]$, the unique solution $x \in [0, \pi]$ of the equation $\cos x = y$ is called *arccosine* or *inverse cosine* of y, denoted $\arccos y$ or $\cos^{-1} y$. This mapping $y \mapsto x$ with $\cos x = y$ has domain $[-1, 1]$ and image $[0, \pi]$. The graph of the arccossine function when drawn in the usual Cartesian plane is symmetric to the graph of $\cos x$ with respect to the line $y = x$.



- The *inverse tangent function* $f(x) = \arctan x$: we restrict the function $\tan x$, which is *not* bijective (not injective) when defined on its usual domain, the set $\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\}$, to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, obtaining thereby a bijective function from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(-\infty, +\infty) = \mathbb{R}$:



Thus given $y \in \mathbb{R}$, the unique solution $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ of the equation $\tan x = y$ is called *arctangent* or *inverse tangent* of y, denoted $\arctan y$ or $\tan^{-1} y$. This mapping $y \mapsto x$ with $\tan x = y$ has domain $\mathbb{R}$ and image $(-\frac{\pi}{2}, \frac{\pi}{2})$. The graph of the arctangent function when drawn in the usual Cartesian plane is symmetric to the graph of $\tan x$ with respect to the line $y = x$.



15.2 Differentiation of an Inverse Function

We begin with a theorem establishing conditions for the existence and continuity of the inverse function.

Theorem 31 (The Inverse Function Theorem)**.** A function $f\colon X \to \mathbb{R}$ that is strictly monotonic on a set $X \subseteq \mathbb{R}$ has an inverse $f^{-1}\colon Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f. The function $f^{-1}\colon Y \to \mathbb{R}$ is monotonic and has the same type of monotonicity on Y that f has on X.

If in addition X is a closed interval $[a, b]$ and f is continuous on X then the set $Y = f(X)$ is the closed interval with endpoints $f(a)$ and $f(b)$ and the function $f^{-1}\colon Y \to X$ is continuous on it.

Proof. **EXTRA CREDIT!!!** ∎

Theorem 32 (The derivative of an inverse function)**.** Let the functions $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ be mutually inverse and continuous at points $x_0 \in X$ and $f(x_0) = y_0 \in Y$ respectively. If f is differentiable at x_0 and $f'(x_0) \neq 0$ then f^{-1} is also differentiable at the point y_0 and

$$\left(f^{-1}(y_0)\right)' = \left(f'(x_0)\right)^{-1} = \frac{1}{f'(x_0)}.$$

Proof. Since the functions $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are mutually inverse, the quantities $f(x) - f(x_0)$ and $f^{-1}(y) - f^{-1}(y_0)$, where $y = f(x)$, are both nonzero if $x \neq x_0$. In addition, we conclude from the continuity of f at x_0 and f^{-1} at y_0 that $(x \to x_0) \Leftrightarrow (y \to y_0)$. Now using the theorem on the limit of a composite function (*explain where and how!*) and the arithmetic properties of the limit, we find

$$(f^{-1}(y_0))' = \lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x\to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}.$$

■

NOTE 36. If we knew in advance that the function f^{-1} was differentiable at y_0 we would find immediately by the identity $(f^{-1} \circ f)(x) = x$ and the Chain Rule that

$$\left.\frac{\mathrm{d}(f^{-1}\circ f)}{\mathrm{d}x}\right|_{x=x_0} = \left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}y}\right|_{y=y_0} \cdot \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \left(f^{-1}\right)'(y_0)\cdot f'(x_0) = 1 \Longrightarrow \left(f^{-1}(y_0)\right)' = \frac{1}{f'(x_0)}.$$

NOTE 37. The condition $f'(x_0) \neq 0$ is equivalent to the statement that the mapping $h \mapsto f'(x_0)h$ realized by the differential $\mathrm{d}_{|_{x=x_0}} f\colon T\mathbb{R}(x_0) \to T\mathbb{R}(y_0)$ has the inverse mapping $[\mathrm{d}_{|_{x=x_0}} f]^{-1}\colon T\mathbb{R}(y_0) \to T\mathbb{R}(x_0)$ given by the formula $\tau \mapsto (f'(x_0))^{-1}\tau$.

Hence in terms of differentials we can write the second statement in Theorem 32 as follows:

If a function f is differentiable at a point x_0 and its differential $\mathrm{d}_{|_{x=x_0}} f\colon T\mathbb{R}(x_0) \to T\mathbb{R}(y_0)$ is invertible at that point then the differential of the function f^{-1} inverse to f exists at the point $y_0 = f(x_0)$ and is the mapping

$$\mathrm{d}_{|_{y=y_0}} f^{-1} = \left[\mathrm{d}_{|_{x=x_0}} f\right]^{-1} : T\mathbb{R}(y_0) \to T\mathbb{R}(x_0),$$

inverse to $\mathrm{d}_{|_{x=x_0}} f\colon T\mathbb{R}(x_0) \to T\mathbb{R}(y_0)$.

15.3 Examples

1. Differentiate $f(x) = \arcsin x$, $x \in [-1, 1]$. Set $y = \arcsin x$ so $x = \sin y$, where $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then

$$f'(x) = y'(x) = \frac{1}{x'(y)} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}.$$

2. Differentiate $f(x) = \arccos x$, $x \in [-1, 1]$. Set $y = \arccos x$ so $x = \cos y$, where $y \in [0, \pi]$. Then

$$f'(x) = y'(x) = \frac{1}{x'(y)} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-\cos^2(\arccos x)}} = -\frac{1}{\sqrt{1-x^2}}.$$

3. Differentiate $f(x) = \arctan x$, $x \in \mathbb{R}$. Set $y = \arctan x$ so $x = \tan y$, where $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$f'(x) = y'(x) = \frac{1}{x'(y)} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{\tan^2(\arctan x) + 1} = \frac{1}{x^2+1}.$$

4. Differentiate $f(x) = \operatorname{arccot} x$, $x \in \mathbb{R}$. Set $y = \operatorname{arccot} x$ so $x = \cot y$, where $y \in (0, \pi)$. Then

$$f'(x) = y'(x) = \frac{1}{x'(y)} = \frac{1}{-\csc^2 y} = -\frac{1}{\cot^2 y + 1} = -\frac{1}{\cot^2(\operatorname{arccot} x) + 1} = -\frac{1}{x^2+1}.$$

15.4 Problems

1.

16 Lecture 15 — Classical Theorems of Differential Calculus

16.1 Fermat's Theorem and Rolle's Theorem

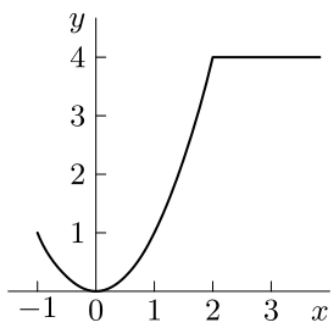
DEFINITION 48 (Local Maximum/Minimum)**.** A point $x_0 \in D \subseteq \mathbb{R}$ is called a *local maximum* (resp. *local minimum*) and the value of a function $f\colon D \to \mathbb{R}$ at that point a *local maximum value* (resp. *local minimum value*) if there exists a neighborhood $U_D(x_0)$ of x_0 in D such that at any point $x \in U_D(x_0)$ we have $f(x) \leqslant f(x_0)$ (resp. $f(x) \geqslant f(x_0)$).

DEFINITION 49 (Strict Local Maximum/Minimum)**.** If the strict inequality $f(x) < f(x_0)$ (resp. $f(x) > f(x_0)$) holds at every point $x_0 \in U_D(x_0) \setminus \{x_0\}$, the point x_0 is called a *strict local maximum* (resp. *lstrict ocal minimum*) and the value of a function $f\colon D \to \mathbb{R}$ at that point a *strict local maximum value* (resp. *strict local minimum value*).

DEFINITION 50. The local maxima and minima are called *local extrema* and the values of the function at these points *local extreme values* of the function.

Example 61. Let

$$f(x) = \begin{cases} x^2, & \text{if } -1 \leqslant x < 2, \\ 4, & \text{if } 2 \leqslant x. \end{cases}$$



For this function

- $x = -1$ is a strict local maximum;
- $x = 0$ is a strict local minimum;
- $x = 2$ is a local maximum;
- the points $x > 2$ are local extrema, being simultaneously maxima and minima, since the function is locally constant at these points.

DEFINITION 51. An extremum $x_0 \in D$ of the function $f\colon D \to \mathbb{R}$ is called an *interior extremum* if x_0 is a limit point of both sets $\{x \in D \mid x < x_0\}$ and $\{x \in D \mid x > x_0\}$.

In Example 61, the points $x = 0$ and $x = 2$ are interior extrema, while the point $x = -1$ is not.

Theorem 33 (Fermat)**.** If a function $f\colon D \to \mathbb{R}$ is differentiable at an interior extremum $x_0 \in D$ then its derivative at x_0 is 0: $f'(x_0) = 0$.

Proof. By definition of differentiability at x_0 we have

$$f(x_0 + h) - f(x_0) = f'(x_0)h + \alpha(x_0; h)h,$$

where $\alpha(x_0; h) \to 0$ as $h \to 0$, $x + h \in D$.

Let us rewrite this relation as follows:

$$f(x_0 + h) - f(x_0) = [f'(x_0) + \alpha(x_0; h)]\, h. \tag{13}$$

Since x_0 is an extremum, the left-hand side of (13) is either nonnegative or nonpositive for all values of h sufficiently close to 0 and for which $x_0 + h \in D$.

If $f'(x_0) \neq 0$ then for h sufficiently close to 0 the quantity $f'(x_0) + \alpha(x_0; h)$ would have the same sign as $f'(x_0)$ since $\alpha(x_0; h) \to 0$ as $h \to 0$, $x_0 + h \in D$.

But the value of h can be both positive or negative given that x_0 is an *interior* extremum.

Thus assuming that $f'(x_0) \neq 0$ we find that the right-hand side of (13) changes sign when h does (for h sufficiently close to 0), while the left-hand side cannot change sign when h is sufficiently close to 0. This contradiction completes the proof. ■

NOTE 38. The converse to Fermat's Theorem is generally false, i.e. $f'(x_0) = 0$ generally does not imply that x_0 is a local extremum.

NOTE 39. Geometrically this theorem asserts that at an extremum of a differentiable function the tangent to its graph is horizontal. (After all, $f'(x_0)$ is the tangent of the angle the tangent line makes with the x-axis.)

Theorem 34 (Rolle's theorem)**.** If a function $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$ then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof. Since the function f is continuous on $[a, b]$, there exist points $x_m, x_M \in [a, b]$ at which it assumes its minimal and maximal values respectively. If $f(x_m) = f(x_M)$ then the function is constant on $[a, b]$; and since in that case $f'(x) \equiv 0$, the assertion follows. If $f(x_m) < f(x_M)$ then since $f(a) = f(b)$, one of the points x_m and x_M must lie in the open interval (a, b). We denote it by ξ. Fermat's Theorem now implies that $f'(\xi) = 0$. ■

16.2 The Theorems of Lagrange and Cauchy on Finite Increments

The following proposition is one of the most frequently used and important methods of studying numerical-valued functions.

Theorem 35 (Lagrange's finite-increment theorem)**.** If a function $f\colon [a,b] \to \mathbb{R}$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) then there exists a point $\xi \in (a,b)$ such that

$$f(b) - f(a) = f'(\xi)(b-a).$$

NOTE 40. This theorem is often referred to as the *Mean Value Theorem* or simply *MVT*.

Proof. Consider the auxiliary function

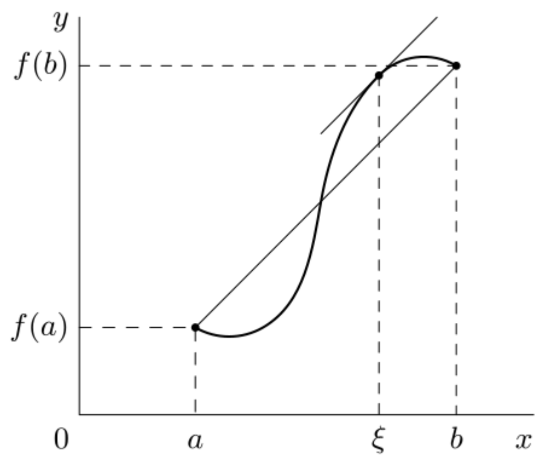
$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a),$$

which is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) and has equal values at the endpoints: $F(a) = F(b) = f(a)$. Applying Rolle's theorem to $F(x)$, we find a point $\xi \in (a,b)$ at which

$$F'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a} = 0.$$

■

NOTE 41. In geometric language Lagrange's theorem means that at some point $(\xi, f(\xi))$, where $\xi \in (a,b)$, the tangent to the graph of the function is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$, since the slope of the chord equals $\frac{f(b)-f(a)}{b-a}$.



Here is a simply nice application of Lagrange's theorem.

COROLLARY 10. A function that is continuous on a closed interval $[a,b]$ is constant on it if and only if its derivative equals zero at every point of the interval $[a,b]$ (or only the open interval (a,b)).

COROLLARY 11. If the derivatives $F_1'(x)$ and $F_2'(x)$ of two functions $F_1(x)$ and $F_2(x)$ are equal on some interval, that is, $F_1'(x) = F_2'(x)$ on the interval then the difference $F_1(x) - F_2(x)$ is constant.

The following proposition is a useful generalization of Lagrange's theorem, and is also based on Rolle's theorem.

Theorem 36 (Cauchy's finite-increment theorem)**.** Let $x = x(t)$ and $y = y(t)$ be functions that are continuous on a closed interval $[\alpha, \beta]$ and differentiable on the optn interval (α, β) .

Then there exists a point $\tau \in (\alpha, \beta)$ such that

$$x'(\tau)(y(\beta) - y(\alpha)) = y'(\tau)(x(\beta) - x(\alpha)).$$

If in addition $x'(t) \neq 0$ for each $t \in (\alpha, \beta)$ then $x(\alpha) \neq x(\beta)$ and we have the equality

$$\frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}.$$

NOTE 42. If $x(\tau) = \tau$ we obtain Lagrange's theorem 35.

Proof. Set $F(t) = x(t)(y(\beta) - y(\alpha)) - y(t)(x(\beta) - x(\alpha))$ for $t \in [\alpha, \beta]$ and apply Rolle's theorem. The rest follows easily. ∎

16.3 Taylor's Formula Extra credit!

16.4 Problems

1.

17 Lecture 16 — Applications of Differentiation

17.1 Conditions for a Function to be Monotonic

Theorem 37. The following relations hold between the monotonicity properties of a function $f\colon D \to \mathbb{R}$ that is differentiable on an open interval $(a, b) = D$ and the sign (positivity) of its derivative f' on that interval:

$$\begin{aligned} f'(x) > 0 \Rightarrow & \quad f \text{ is increasing} & \Rightarrow f'(x) \geqslant 0, \\ f'(x) \geqslant 0 \Rightarrow & \quad f \text{ is nondecreasing} & \Rightarrow f'(x) \geqslant 0, \\ f'(x) \equiv 0 \Rightarrow & \quad f \equiv \text{const.} & \Rightarrow f'(x) \equiv 0, \\ f'(x) < 0 \Rightarrow & \quad f \text{ is decreasing} & \Rightarrow f'(x) \leqslant 0. \end{aligned}$$

Proof. The left-hand column of implications follows immediately from Lagrange's theorem, by virtue of which $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$, where $x_1, x_2 \in (a, b)$ and ξ is a point between x_1 and x_2. It can be seen from this formula that for $x_1 < x_2$ the difference $f(x_2) - f(x_1)$ is positive if and only if $f'(\xi)$ is positive.

The right-hand column of implications can be obtained from the definition of the derivative. Let us show, for example, that if a function f is differentiable on (a, b) is increasing then $f'(x) \geqslant 0$ on (a, b). Indeed,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If $h > 0$ then $f(x + h) - f(x) > 0$; and if $h < 0$ then $f(x + h) - f(x) < 0$. Therefore the fraction after the limit sign is positive.

Consequently, its limit $f'(x)$ is nonnegative, as asserted. ■

NOTE 43. It is clear from the example of the function $f(x) = x^3$ that a strictly increasing function has a nonnegative derivative, not necessarily one that is always positive. In this example, $f'(0) = 3x^2|_{x=0} = 0$.

17.2 Conditions for an Interior Extremum of a Function

Taking account of Fermat's theorem, we can state the following result.

Theorem 38. In order for a point x_0 to be an extremum of a function $f\colon U(x_0) \to \mathbb{R}$ defined on a neighborhood $U(x_0)$ of that point, a necessary condition is that one of the following two conditions hold: either the function is not differentiable at x_0 or $f'(x_0) = 0$.

Simple examples show that these necessary conditions are not sufficient.

Example 62. Let $f(x) = x^3$ on $\mathbb{R}$. Then $f'(0) = 0$ but there is no extremum at $x_0 = 0$.

Example 63. Let

$$f(x) = \begin{cases} x, & \text{for } x > 0, \\ 2x, & \text{for } x < 0. \end{cases}$$

This function has a bend at 0 and obviously has neither a derivative nor an extremum at 0.

Theorem 39. Let $f\colon U(x_0) \to \mathbb{R}$ be a function defined on a neighborhood $U(x_0)$ of the point x_0, which is continuous at the point itself and differentiable in a deleted neighborhood $\mathring{U}(x_0)$. Let $\mathring{U}^-(x_0) = \{x \in U(x_0) \mid x < x_0\}$ and $\mathring{U}^+(x_0) = \{x \in U(x_0) \mid x > x_0\}$.

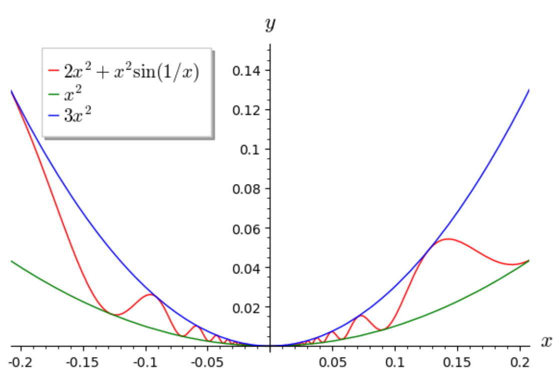
Then the following conclusions are valid:

1. $(\forall x \in \mathring{U}^-(x_0)\ (f'(x) < 0)) \wedge (\forall x \in \mathring{U}^+(x_0)\ (f'(x) < 0)) \Rightarrow (f$ has no extremum at $x_0)$;
2. $(\forall x \in \mathring{U}^-(x_0)\ (f'(x) < 0)) \wedge (\forall x \in \mathring{U}^+(x_0)\ (f'(x) > 0)) \Rightarrow (x_0$ is a strict local minimum of $f)$;
3. $(\forall x \in \mathring{U}^-(x_0)\ (f'(x) > 0)) \wedge (\forall x \in \mathring{U}^+(x_0)\ (f'(x) < 0)) \Rightarrow (x_0$ is a strict local maximum of $f)$;
4. $(\forall x \in \mathring{U}^-(x_0)\ (f'(x) > 0)) \wedge (\forall x \in \mathring{U}^+(x_0)\ (f'(x) > 0)) \Rightarrow (f$ has no extremum at $x_0)$.

NOTE 44. Briefly, but less precisely, one can say that if the derivative changes sign in passing through the point then the point is an extremum, while if the derivative does not change sign the point is not an extremum. The following example shows that these conditions are not necessary.

Example 64. Let

$$f(x) = \begin{cases} 2x^2 + x^2 \sin\frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$



Since $x^2 \leqslant f(x) \leqslant 3x^2$ for all $x \in R$, it is clear that the function has a strict local minimum at $x_0 = 0$ but the derivative $f'(x) = 4x + 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ is not of constant sign in any deleted one-sided neighborhood of this point.

Proof. (1) It follows from Theorem 37 that f is strictly decreasing on $\mathring{U}^-(x_0)$. That is, for any $x, y \in \mathring{U}^-(x_0)$, $x < y$ implies $f(x) > f(y)$. Passing to limit as $\mathring{U}^-(x_0) \ni y \to x_0$ we see that $f(x) \geqslant \lim_{\mathring{U}^-(x_0)\ni y\to x_0} f(y) = f(x_0)$. Clearly $f(x) = f(x_0)$ is impossible since then $f(y) = f(x_0)$ for all $y \in (x, x_0)$ violating the strict monotonicity of f on $\mathring{U}^-(x_0)$. Hence $f(x) > f(x_0)$ for any $x \in \mathring{U}^-(x_0)$. By the same considerations we have $f(x_0) > f(x)$ for all $x \in \mathring{U}^+(x_0)$. Thus the function is strictly decreasing in the whole neighborhood $U(x_0)$ and x_0 is not an extremum.

(2) We conclude as in (1) that since $f(x)$ is decreasing on $\mathring{U}^-(x_0)$ and continuous at x_0, we have $f(x) > f(x_0)$ for all $x \in \mathring{U}^-(x_0)$. We then argue that from the increasing nature of f on $\mathring{U}^+(x_0)$ we have $f(x_0) < f(x)$ for all $x \in \mathring{U}^+(x_0)$. Thus f has a strict local minimum at x_0.

Statements (3) and (4) are proved similarly. ■

Theorem 40 (Sufficient conditions for an extremum in terms of higher-order derivatives)**. Extra credit!**

17.3 Conditions for a Function to be Convex Extra creidt

17.4 L'Hôpital's Rule

L'Hôpital's rule is a useful result (or rather a number of related results) that allows us to compute some rather complicated limits using derivatives.

Zorich give a very generic statement and proof which lacks some details. The proof on Wikipedia might be a little too complex. Consider using 's approach. The unified proof that Zorich botched and to which Wikipedia refers is offered for extra credit.

Theorem 41 ($\frac{0}{0}$ at a point)**.** Let functions $f(x)$ and $g(x)$ be defined on $(a,b]$ and

1. $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} f(x) = 0$,
2. both f and g are differentiable on (a,b) and $g'(x) \neq 0$ for any $x \in (a,b)$, and
3. there exists (finite or infinite) limit
$$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = K.$$

Then
$$\lim_{x\to a+} \frac{f(x)}{g(x)} = K.$$

Proof. The proof relies on Cauchy's finite increment theorem. Define f and g at a by setting $f(a) = g(a) = 0$. Then these functions are continuous on $[a,t]$ for any $t \in (a,b)$: their values at $x = a$ equal to their limits at $x = a$ and at all other points their continuity follows from their differentiability. Applying now Cauchy's theorem on the interval $[a,t]$ for any $t \in (a,b)$ we have
$$\frac{f(t)}{g(t)} = \frac{f(t)-f(a)}{g(t)-g(a)} = \frac{f'(c)}{g'(c)}, \tag{14}$$
where $a < c < t$. We note that $g(t) = g(t) - g(a) \neq 0$ since otherwise by Rolle's theorem we would have $g'(x) = 0$ for some $x \in (a,t)$.

Now sending $t \to a+$ in (14) we have $c \to a+$, and
$$\lim_{t\to a+} \frac{f(t)}{g(t)} = \lim_{c\to a+} \frac{f'(c)}{g'(c)} = K.$$
■

Go to Example 1.

Theorem 42 ($\frac{0}{0}$ at infinity)**.** Let functions $f(x)$ and $g(x)$ be defined on $[c,+\infty)$, $c > 0$, and

1. $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 0$,
2. both f and g are differentiable on $(c,+\infty)$ and $g'(x) \neq 0$ for any $x \in (c,+\infty)$, and
3. there exists (finite or infinite) limit
$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = K.$$

Then
$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = K.$$

Proof. Changing variables we set $x = \frac{1}{t}$ and introduce functions $F(t) = f(1/t)$ and $G(t) = g(1/t)$. The functions F and G are defined on $(0, \frac{1}{c}]$ and differentiable on $(0, \frac{1}{c})$. Also $x \to +\infty$ implies $t \to +0$, and

$$\lim_{t\to+0} F(t) = \lim_{x\to+\infty} f(x) = 0, \quad \lim_{t\to+0} G(t) = \lim_{x\to+\infty} g(x) = 0,$$

and

$$\lim_{t\to+0} \frac{F'(t)}{G'(t)} = \lim_{t\to+0} \frac{f'(t)\cdot\left(-\frac{1}{t^2}\right)}{g'(t)\cdot\left(-\frac{1}{t^2}\right)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = K.$$

We now apply Theorem 41 to functions F and G

$$\lim_{t\to+0} \frac{F(t)}{G(t)} = \lim_{t\to+0} \frac{F'(t)}{G'(t)} = K.$$

■

17.5 Examples

1. Compute the limit
$$\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}.$$

2. Compute the limit
$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - \mathrm{e}}{x}.$$

17.6 Problems

1.