Suffolk County Community College
Michael J. Grant Campus
Department of Mathematics

Monday, December 17, 2018

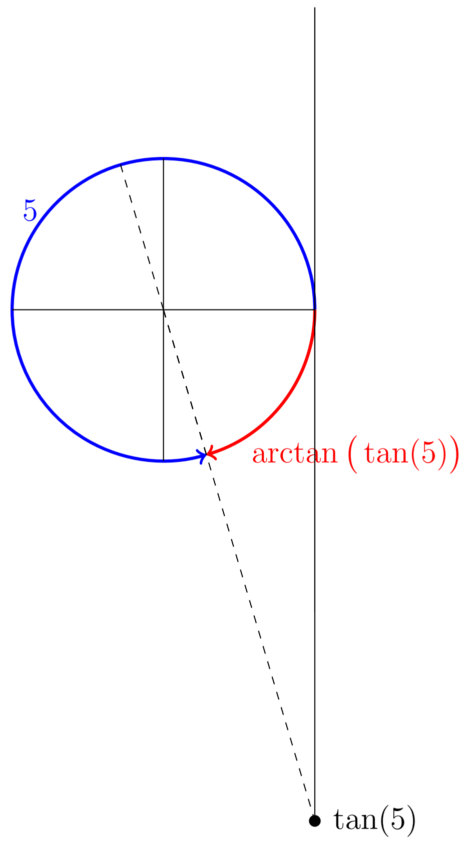
MAT 125: Pre-Calculus II

Final Exam: Solutions and Answers

Instructor:
Name: Alexander Kasiukov
Office: Health, Science and Education Center, Room 109
Phone: (631) 851-6484
Email: kasiuka@sunysuffolk.edu
Web Site: `http://www.kasiukov.com`

Problem 1. Consider the expression $\arctan\big(\tan(5)\big)$. Draw 5, $\tan(5)$ and $\arctan\big(\tan(5)\big)$ in the same picture, showing how they are interconnected.

Space for your solution:



Problem 2. Use the above picture to express $\arctan\big(\tan(5)\big)$ without any trigonometric functions.

Space for your solution:

It is clear from the above picture that

$$5 - \arctan\big(\tan(5)\big) = 2\pi,$$

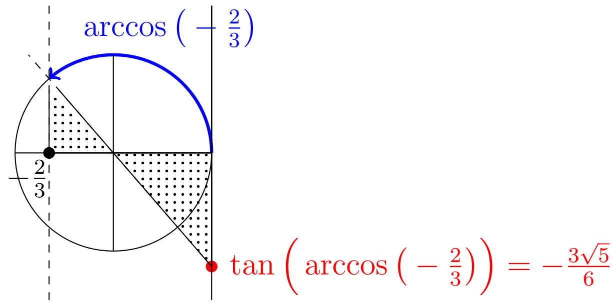
therefore

$$\arctan\big(\tan(5)) = 5 - 2\pi.$$

Problem 3. Draw the elements of $\tan\Big(\arccos\big(-\frac{2}{3}\big)\Big)$ to express it without trigonometric functions.

Space for your solution:

Similarity of the two shaded triangles:



implies proportionality: $\dfrac{\tan\Big(\arccos\big(-\frac{2}{3}\big)\Big)}{1} = \dfrac{\sin\Big(\arccos\big(-\frac{2}{3}\big)\Big)}{\big(-\frac{2}{3}\big)} \quad \Leftrightarrow$

$$\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = -\frac{3\sin\Big(\arccos\big(-\frac{2}{3}\big)\Big)}{2} =$$

$$= \boxed{\text{Pythagorean identity and the fact that the sin we are looking for is non-negative}} =$$

$$= -\frac{3\sqrt{1-\big(-\frac{2}{3}\big)^2}}{2} = -\frac{3\sqrt{\frac{5}{9}}}{2} = -\frac{3\sqrt{5}}{6}.$$

Problem 4. Express $\tan\Big(\arccos(x)\Big)$ without trigonometric functions and determine for which x this expression makes sense.

Space for your solution:

Generalizing the above consideration for any $x \in [-1, 1]$, we get:

$$\tan\Big(\arccos(x)\Big) = \frac{\sin\Big(\arccos(x)\Big)}{\cos\Big(\arccos(x)\Big)} = \frac{\sin\Big(\arccos(x)\Big)}{x} =$$

$$= \boxed{\text{Pythagorean identity and the fact that the sin we are looking for is non-negative}} =$$

$$= \frac{\sqrt{1-\cos^2\Big(\arccos(x)\Big)}}{x} = \frac{\sqrt{1-x^2}}{x}.$$

Problem 5. Solve the equation $\sin(2t) = \tan(t)$.

Space for your solution:

$\sin(2t) = \tan(t) \Leftarrow$ make all arguments the same using $\sin(2t) = 2\sin(t)\cos(t)$ $\Rightarrow$

$2\sin(t)\cos(t) = \tan(t) \Leftarrow$ get rid of tan using $\tan(t) = \frac{\sin(t)}{\cos(t)}$ $\Rightarrow$ $2\sin(t)\cos(t) = \frac{\sin(t)}{\cos(t)}$

$\Leftarrow$ multiply both sides by $\cos(t)$ $\Rightarrow$ $2\sin(t)\cos(t)\cos(t) = \frac{\sin(t)}{\cos(t)}\cos(t)$

$\Leftarrow$ cancel the denominator, but remember it is nonzero $\Rightarrow$ $\begin{cases} 2\sin(t)\cos^2(t) = \sin(t) \\ \cos(t) \neq 0 \end{cases}$

$\Leftarrow$ divide both sides by $\sin(t)$, but consider separately the case when it is zero $\Rightarrow$

$$\begin{cases} \left[\begin{array}{l} \sin(t) = 0 \\ \begin{cases} 2\cos^2(t) = 1 \\ \sin(t) \neq 0 \end{cases} \end{array} \right. \\ \cos(t) \neq 0 \end{cases}$$

$\Leftarrow$ apply De Morgan's law $\Rightarrow$

$$\left[\begin{array}{l} \begin{cases} \sin(t) = 0 \\ \cos(t) \neq 0 \end{cases} \\ \begin{cases} 2\cos^2(t) = 1 \\ \sin(t) \neq 0 \\ \cos(t) \neq 0 \end{cases} \end{array} \right.$$

$\Leftarrow$ remove redundant restrictions $\Rightarrow$ $\left[\begin{array}{l} \sin(t) = 0 \\ \cos^2(t) = \frac{1}{2} \end{array} \right.$ $\Leftarrow$ solve simple quadratic equation $\Rightarrow$

$$\left[\begin{array}{l} \sin(t) = 0 \\ \left[\begin{array}{l} \cos(t) = \frac{\sqrt{2}}{2} \\ \cos(t) = -\frac{\sqrt{2}}{2} \end{array} \right. \end{array} \right. \Leftrightarrow \left[\begin{array}{l} \sin(t) = 0 \\ \cos(t) = \frac{\sqrt{2}}{2} \\ \cos(t) = -\frac{\sqrt{2}}{2} \end{array} \right. \Leftrightarrow \left[\begin{array}{l} t = \pi n; n \in \mathbb{Z} \\ t = \pm \arccos\left(\frac{\sqrt{2}}{2}\right) + 2\pi n; n \in \mathbb{Z} \\ t = \pm \arccos\left(-\frac{\sqrt{2}}{2}\right) + 2\pi n; n \in \mathbb{Z} \end{array} \right.$$

$$\Leftrightarrow \left[\begin{array}{l} t = \pi n; n \in \mathbb{Z} \\ t = \pm\frac{\pi}{4} + 2\pi n; n \in \mathbb{Z} \\ t = \pm\frac{3\pi}{4} + 2\pi n; n \in \mathbb{Z} \end{array} \right.$$

$\Leftarrow$ (optionally) write solutions in a more compact way $\Rightarrow$

$$\left[\begin{array}{l} t = \pi n; n \in \mathbb{Z} \\ t = \frac{\pi}{4} + \frac{\pi}{2}n; n \in \mathbb{Z} \end{array} \right.$$

Problem 6. Find all complex numbers z, such that $z^3 = -i$. (You may use polar coordinates, but the final answer must explicitly give the $\mathrm{Re}(z)$ and $\mathtt{Im}\ (z)$.) Graph all solutions on the (x, y) plane.

Space for your solution:

$$z^3 = -i \quad \Leftrightarrow \quad \begin{cases} |z^3| = |-i| \\ \arg(z^3) = \arg(-i) + 2\pi n, \ n \in \mathbb{Z} \end{cases} \quad \Leftrightarrow \quad \begin{cases} |z|^3 = 1 \\ 3 \cdot \arg(z) = -\frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z} \end{cases}$$

$$\Leftrightarrow \begin{cases} |z| = 1 \\ \arg(z) = -\frac{\pi}{6} + \frac{2\pi n}{3}, \ n \in \mathbb{Z} \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{l} z = \cos(-\frac{\pi}{6}) + \sin(-\frac{\pi}{6})i \\ z = \cos(\frac{3\pi}{6}) + \sin(\frac{3\pi}{6})i \\ z = \cos(\frac{7\pi}{6}) + \sin(\frac{7\pi}{6})i \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} z = \frac{\sqrt{3}}{2} - \frac{1}{2}i \\ z = i \\ z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \end{array} \right.$$