Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 125: Pre-Calculus II

Final Exam: Solutions and Answers

Instructor:

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Problem 1. Suppose an angle $\theta \in [0, \pi]$ and $\tan(\theta) = -2$. Find $\cos(\theta)$.

Space for your solution:

By definition, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. Since we know $\tan(\theta)$ and are trying to find $\cos(\theta)$, we should express $\sin(\theta)$ in terms of these two functions. Solving the Pythagorean identity $\big(\cos(\theta)\big)^2 + \big(\sin(\theta)\big)^2 = 1$ for the $\sin(\theta)$, we get $\sin(\theta) = \pm\sqrt{1 - \big((\cos(\theta)\big)^2}$. The given fact that $\theta \in [0, \pi]$ implies that the $\sin(\theta)$ is positive: $\sin(\theta) = \sqrt{1 - \big((\cos(\theta)\big)^2}$. Substituting this expression for $\sin(\theta)$ into the definition of $\tan(\theta)$, we get

$$\tan(\theta) = \frac{\sqrt{1 - \big((\cos(\theta)\big)^2}}{\cos(\theta)}.$$

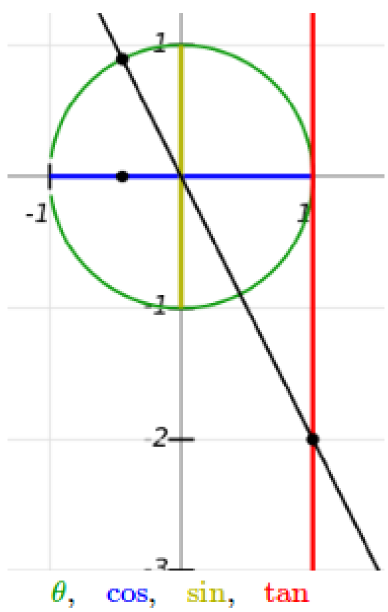
Substituting the known value of $\tan(\theta) = -2$, we get the following equation for $\cos(\theta)$:

$$-2 = \frac{\sqrt{1 - \big((\cos(\theta)\big)^2}}{\cos(\theta)}$$

$$\Leftrightarrow \quad -2\cos(\theta) = \sqrt{1 - \big((\cos(\theta)\big)^2} \quad \Leftrightarrow \quad \begin{cases} 4\big(\cos(\theta)\big)^2 = 1 - \big((\cos(\theta)\big)^2 \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow$$

$$\begin{cases} 5\big(\cos(\theta)\big)^2 = 1 \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \big(\cos(\theta)\big)^2 = \frac{1}{5} \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow \quad \cos(\theta) = -\sqrt{\tfrac{1}{5}} = -\tfrac{1}{\sqrt{5}}.$$

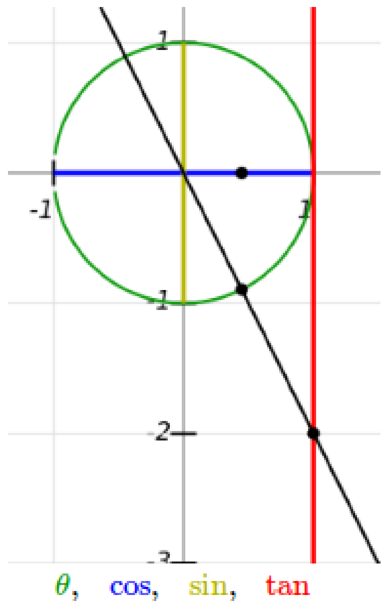
The following picture demonstrates the objects under consideration and suggests another approach to solving this problem — the one based on similararity of triangles.



Problem 2. Find $\cos\big(\arctan(-2)\big)$. More specifically, find an expression of this quantity that does not use any trigonometric functions.

Space for your solution:

This problem is just a slight variation of the previous one. The only issue to be mindful of is the fact that the angle $\arctan(-2)$, unlike the angle θ in the previous problem, must be in the open interval $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ and hence we are looking for the x-coordinate of the other point of intersection of the tilted line with the unit circle.



The computations of the previous problem apply, except for the sign, which needs to be changed:

$$\cos\big(\arctan(-2)\big) = \frac{1}{\sqrt{5}}.$$

Problem 3. For an arbitrary real number y, find $\cos\big(\arctan(y)\big)$. More specifically, find an expression of this quantity that does not use any trigonometric functions.

Space for your solution:

This is a generalization of the previous problem. Denote $x = \cos\big(\arctan(y)\big)$. Then, from the similarity of triangles in the above picture, we get the equation: $y = \pm\frac{\sqrt{1-x^2}}{x} \Leftrightarrow y \cdot x = \pm\sqrt{1-x^2} \Leftrightarrow x^2 \cdot y^2 = 1 - x^2. \Leftrightarrow x^2(y^2+1) = 1 \Leftrightarrow x^2 = \frac{1}{y^2+1} \Leftrightarrow x = \pm\sqrt{\frac{1}{y^2+1}}$ $\Leftarrow$ $\boxed{\text{Since } \arctan(y) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)\text{, we know that } x > 0.}$ $\Rightarrow x = \sqrt{\frac{1}{y^2+1}} = \frac{1}{\sqrt{y^2+1}}.$

Problem 4. Solve the equation $\cot(t) = \sin(t)$.

Space for your solution:

$\cot(t) = \sin(t) \Leftarrow\!\boxed{\text{definition of } \cot(t)}\!\Rightarrow \frac{\cos(t)}{\sin(t)} = \sin(t)$

$\Leftarrow\!\boxed{\text{multiply both sides by } \sin(t)\text{, but remember that it cannot be zero}}\!\Rightarrow$

$\begin{cases} \cos(t) = \big(\sin(t)\big)^2 \\ \sin(t) \neq 0 \end{cases} \Leftarrow\!\boxed{\text{Pythagorean identity}}\!\Rightarrow \begin{cases} \cos(t) = 1 - \big(\cos(t)\big)^2 \\ \sin(t) \neq 0 \end{cases} \Leftrightarrow$

$\begin{cases} \big(\cos(t)\big)^2 + \cos(t) - 1 = 0 \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{cases} \Leftarrow\!\boxed{\text{Quadratic formula}}\!\Rightarrow \begin{cases} \cos(t) = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{cases} \Leftrightarrow$

$\begin{cases} \cos(t) = \frac{-1 \pm \sqrt{5}}{2} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{cases} \Leftarrow\!\boxed{\text{Cosine cannot equal } \frac{-1-\sqrt{5}}{2} < -1.}\!\Rightarrow \begin{cases} \cos(t) = \frac{-1+\sqrt{5}}{2} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{cases} \Leftrightarrow$

$t = \pm \arccos\left(\frac{-1+\sqrt{5}}{2}\right) + 2 \cdot \pi \cdot n, n \in \mathbb{Z}.$

Problem 5. In the triangle $\triangle ABC$, the sides $|AB| = 3$, $|AC| = 4$ and the angle $\widehat{BAC} = \frac{\pi}{8}$.

(1). Find the length $|BC|$.

Space for your solution:

Using the cosine formula,

$$|BC| = \sqrt{|AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos \widehat{BAC}}$$

$$= \sqrt{3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos\left(\frac{\pi}{8}\right)} = \sqrt{9 + 16 - 24 \cdot \cos\left(\frac{45^\circ}{2}\right)}$$

$= \boxed{\text{solving } \cos(2t) = 2\cos^2(t) - 1 \text{ for } \cos(t) \text{ when } t = \frac{45^\circ}{2}\text{, we get } \cos\left(\frac{45^\circ}{2}\right) = \sqrt{\frac{1+\cos(45^\circ)}{2}}} =$

$$= \sqrt{25 - 24\sqrt{\frac{1 + \cos(45^\circ)}{2}}}$$

$$= \sqrt{25 - 24\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}}$$

$$= \sqrt{25 - 24\sqrt{\frac{2 + \sqrt{2}}{4}}} = \sqrt{25 - 12\sqrt{2 + \sqrt{2}}}.$$

(2). Find the angle $\widehat{ABC}$.

Space for your solution:

Using again the cosine formula, we get

$$
\begin{aligned}
\widehat{ABC} &= \arccos\left(\frac{|AB|^2+|BC|^2-|AC|^2}{2\cdot|AB|\cdot|BC|}\right)\\
&= \arccos\left(\frac{3^2+\left(\sqrt{25-12\sqrt{2+\sqrt{2}}}\right)^2-4^2}{2\cdot 3\cdot\sqrt{25-12\sqrt{2+\sqrt{2}}}}\right)\\
&= \arccos\left(\frac{9+25-12\sqrt{2+\sqrt{2}}-16}{6\cdot\sqrt{25-12\sqrt{2+\sqrt{2}}}}\right)\\
&= \arccos\left(\frac{18-12\sqrt{2+\sqrt{2}}}{6\cdot\sqrt{25-12\sqrt{2+\sqrt{2}}}}\right)\\
&= \arccos\left(\frac{3-2\sqrt{2+\sqrt{2}}}{\sqrt{25-12\sqrt{2+\sqrt{2}}}}\right).
\end{aligned}
$$

(3). Find the area of the triangle.

Space for your solution:

Using the sine formula,

$$
\begin{aligned}
\texttt{Area}(\triangle ABC) &= \frac{1}{2}\cdot|AB|\cdot|AC|\cdot\sin\widehat{BAC}\\
&= \frac{1}{2}\cdot 3\cdot 4\cdot\sin\frac{\pi}{8} = 6\cdot\sin\left(\frac{45^\circ}{2}\right)\\
&= 6\sqrt{\frac{1-\cos 45^\circ}{2}} = 6\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = 6\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{2}} = 6\sqrt{\frac{2-\sqrt{2}}{4}}\\
&= 3\sqrt{2-\sqrt{2}}.
\end{aligned}
$$

Problem 6. Find all complex numbers z, such that $z^3 = -27$. (You may use polar coordinates, but the final answer must explicitly give the $\mathrm{Re}(z)$ and $\mathtt{Im}\ (z)$.) Graph all solutions on the (x, y) plane.

Space for your solution:

$z^3 = -27$

$\Leftarrow$ complex numbers are equal iff their lengths and arguments are equal $\Rightarrow$

$$\begin{cases} |z^3| = |-27| \\ \arg(z^3) = \arg(-27) \end{cases} \Leftarrow \boxed{\text{compute length and argument of } -27} \Rightarrow \begin{cases} |z^3| = 27 \\ \arg(z^3) = \pi \end{cases}$$

$\Leftarrow$ properties of length and argument of the product of complex numbers $\Rightarrow$

$$\begin{cases} |z|^3 = 27 \\ \exists\ n \in \mathbb{Z} : 3 \cdot \arg(z) + 2\pi n = \pi \end{cases} \Leftrightarrow \begin{cases} |z| = 3 \\ \exists\ n \in \mathbb{Z} : \arg(z) = \frac{\pi}{3} + \frac{2\pi}{3} \cdot n \end{cases}$$

$\Leftarrow$ if we take $\arg(z) \in (-\pi, \pi]$, then $n = 0, 1, -1$ are the only possible values $\Rightarrow$

$$\left[\begin{array}{l} z = 3\Big(\cos\left(\frac{\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3}\right)\Big) \\ z = 3\Big(\cos\left(\pi\right) + i \cdot \sin\left(\pi\right)\Big) \\ z = 3\Big(\cos\left(-\frac{\pi}{3}\right) + i \cdot \sin\left(-\frac{\pi}{3}\right)\Big) \end{array}\right. \Leftrightarrow \left[\begin{array}{l} z = 3\Big(\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\Big) \\ z = 3\Big(-1 + i \cdot 0\Big) \\ z = 3\Big(\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}\Big) \end{array}\right. \Leftrightarrow \left[\begin{array}{l} z = \frac{3}{2} + i \cdot \frac{3\sqrt{3}}{2} \\ z = -3 \\ z = \frac{3}{2} - i \cdot \frac{3\sqrt{3}}{2} \end{array}\right.$$

Here is how the solutions look like on the (x, y)-plane: