Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 125
Pre-Calculus II

Final Exam: Solutions and Answers

Instructor:

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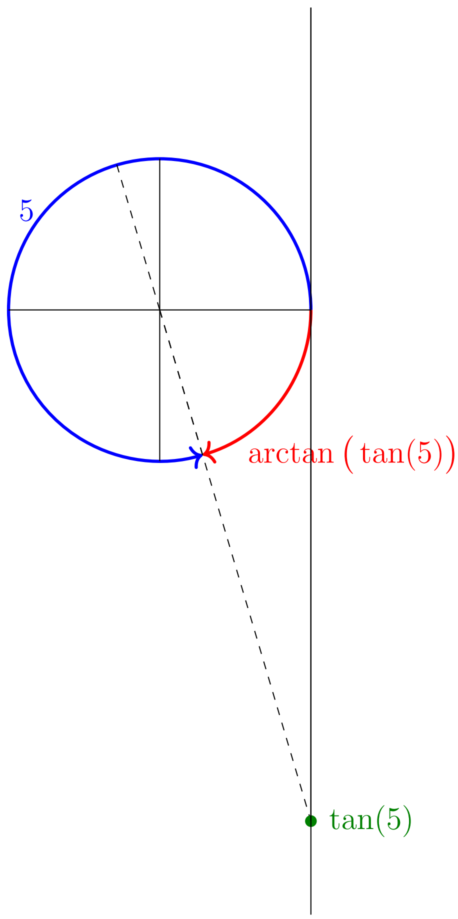
Problem 1. Consider the expression $\arctan\big(\tan(5)\big)$.

(1). Draw 5, $\tan(5)$ and $\arctan\big(\tan(5)\big)$ in the same picture of a unit circle, showing how they are interconnected.

Space for your solution:



(2). Use the above picture to express $\arctan\big(\tan(5)\big)$ without any trigonometric functions.

Space for your solution:

It is clear from the above picture that

$$5 - \arctan\big(\tan(5)\big) = 2\pi,$$

therefore

$$\arctan\big(\tan(5)\big) = 5 - 2\pi.$$

Problem 2. Solve the equation $\cos(t) + \sin(t) = 0$.

Space for your solution:

$$\cos(t) + \sin(t) = 0 \quad \Leftrightarrow \quad \sin(t) = -\cos(t)$$

$$\Leftarrow\!\boxed{\text{divide both sides by } \cos(t) \text{, but remember about the possibility of it being zero}}\!\Rightarrow$$

$$\left[\begin{array}{l} \left\{\begin{array}{l} \cos(t) = 0 \\ \sin(t) = 0 \end{array}\right. \\ \\ \tan(t) = -1 \end{array}\right. \quad \Leftarrow\!\boxed{\text{the subsystem is incompatible}}\!\Rightarrow \quad \tan(t) = -1 \quad \Leftrightarrow$$

$$\exists n \in \mathbb{Z} : t = \arctan(-1) + \pi n \quad \Leftrightarrow \quad \exists n \in \mathbb{Z} : t = -\tfrac{\pi}{4} + \pi n.$$

Problem 3. Solve the equation $\sin(2t) = \tan(t)$.

Space for your solution:

$$\sin(2t) = \tan(t) \Leftarrow\boxed{\text{make all arguments the same using } \sin(2t) = 2\sin(t)\cos(t)}\Rightarrow$$

$$2\sin(t)\cos(t) = \tan(t) \Leftarrow\boxed{\text{get rid of tan using } \tan(t) = \tfrac{\sin(t)}{\cos(t)}}\Rightarrow 2\sin(t)\cos(t) = \tfrac{\sin(t)}{\cos(t)}$$

$$\Leftarrow\boxed{\text{multiply both sides by } \cos(t)}\Rightarrow 2\sin(t)\cos(t)\cos(t) = \tfrac{\sin(t)}{\cos(t)}\cos(t)$$

$$\Leftarrow\boxed{\text{cancel the denominator, but remember it is nonzero}}\Rightarrow \begin{cases} 2\sin(t)\cos^2(t) = \sin(t) \\ \cos(t) \neq 0 \end{cases}$$

$$\Leftarrow\boxed{\text{divide both sides by } \sin(t)\text{, but consider separately the case when it is zero}}\Rightarrow$$

$$\begin{cases} \left[\begin{array}{l} \sin(t) = 0 \\ \begin{cases} 2\cos^2(t) = 1 \\ \sin(t) \neq 0 \end{cases} \end{array}\right. \\ \cos(t) \neq 0 \end{cases} \Leftarrow\boxed{\text{apply De Morgan's law}}\Rightarrow \left[\begin{array}{l} \begin{cases} \sin(t) = 0 \\ \cos(t) \neq 0 \end{cases} \\ \begin{cases} 2\cos^2(t) = 1 \\ \sin(t) \neq 0 \\ \cos(t) \neq 0 \end{cases} \end{array}\right.$$

$$\Leftarrow\boxed{\text{drop redundant restrictions}}\Rightarrow \left[\begin{array}{l} \sin(t) = 0 \\ \cos^2(t) = \tfrac{1}{2} \end{array}\right. \Leftarrow\boxed{\text{solve simple quadratic equation}}\Rightarrow$$

$$\left[\begin{array}{l} \sin(t) = 0 \\ \left[\begin{array}{l} \cos(t) = \tfrac{\sqrt{2}}{2} \\ \cos(t) = -\tfrac{\sqrt{2}}{2} \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \sin(t) = 0 \\ \cos(t) = \tfrac{\sqrt{2}}{2} \\ \cos(t) = -\tfrac{\sqrt{2}}{2} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \exists n \in \mathbb{Z} : t = \pi n \\ \exists n \in \mathbb{Z} : t = \pm\arccos\left(\tfrac{\sqrt{2}}{2}\right) + 2\pi n \\ \exists n \in \mathbb{Z} : t = \pm\arccos\left(-\tfrac{\sqrt{2}}{2}\right) + 2\pi n \end{array}\right.$$

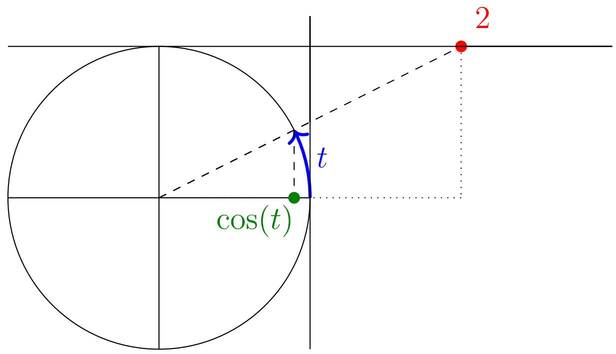
$$\Leftrightarrow \left[\begin{array}{l} \exists n \in \mathbb{Z} : t = \pi n \\ \exists n \in \mathbb{Z} : t = \pm\tfrac{\pi}{4} + 2\pi n \\ \exists n \in \mathbb{Z} : t = \pm\tfrac{3\pi}{4} + 2\pi n \end{array}\right. \Leftarrow\boxed{\text{optional — more compact — form}}\Rightarrow$$

$$\left[\begin{array}{l} \exists n \in \mathbb{Z} : t = \pi n \\ \exists n \in \mathbb{Z} : t = \tfrac{\pi}{4} + \tfrac{\pi}{2}n \end{array}\right.$$

Problem 4. In this problem, we will study $\cos\Big(\operatorname{arccot}(x)\Big)$.

(1). Suppose $t \in [0, \pi]$ and $\cot(t) = 2$. Mark the $2, t$ and $\cos(t)$ in the proper locations in the picture of the unit circle.

Space for your solution:



(2). Use the above picture to express $\cos(t)$ without trigonometric functions.

Space for your solution:

The similarity of the larger (dotted) triangle with small cathetus 1, big cathetus 2 and hypotenuse $\sqrt{1^2 + 2^2} = \sqrt{5}$ [a] — on the one hand, and the smaller (dashed) triangle having small cathetus $\sin(t)$, big cathetus $\cos(t)$ and hypotenuse 1, yields the proportion:

$$\frac{\cos(t)}{1} = \frac{2}{\sqrt{5}}.$$

[a] from the Pythagorean theorem

(3). For all $x \in \mathbb{R}$, express $\cos\Big(\operatorname{arccot}(x)\Big)$ without trigonometric functions.

Space for your solution:

Generalizing the above picture by replacing the 2 with an arbitrary $x \in \mathbb{R}$, we get:

$$\cos\Big(\operatorname{arccot}(x)\Big) = \frac{x}{\sqrt{1 + x^2}}.$$