Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 124
Pre-Calculus I

Final Exam: Solutions and Answers

Instructor:

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Problem 1. Suppose

- set $A = \{\text{flu}, \text{headache}, \text{fever}, \text{allergy}, \text{Lime disease}\}$;
- set $B = \{\text{Aspirin}, \text{Relenza}, \text{Claritin}, \text{Doxycycline}\}$;
- set $C =$

$$\Big\{(\text{flu}, \text{Relenza}), (\text{headache}, \text{Aspirin}), (\text{fever}, \text{Aspirin}), (\text{allergy}, \text{Claritin}), (\text{Lime disease}, \text{Doxycycline})\Big\}.$$

(1). Does the tripple $T = (A, B, C)$ constitute a binary relation? Why?

Space for your solution:

Yes, the tripple $T = (A, B, C)$ constitutes a binary relation because $C \subseteq A \times B$.

(2). Does the tripple $T = (A, B, C)$ constitute a function? Why?

Space for your solution:

Yes, the tripple $T = (A, B, C)$ constitutes a function. First, we verified in the previous sub-problem that T is a binary relation. Further, for every medical condition in A there is one and only one medication in B which is linked to that condition by being in C. Thus T is a binary relation that passes the vertical line test, which is to say T is a function.

(3). What is $\mathtt{T}$(headache)?

Space for your solution:

By definition, $\mathtt{T}$(headache) is the second element of the only ordered pair in the graph C of the function $\mathtt{T}$ that has "headache" as its first element. In our case, such a pair is the pair (headache, Aspirin), therefore $\mathtt{T}$(headache) = Aspirin.

(4). What is the image of $\mathtt{T}$?

Space for your solution:

The question makes sense because $\mathtt{T}$ is a function. By definition,

$$\mathtt{Im}\ \mathtt{T} = \{\mathtt{T}(x) : x \in \mathtt{Dom}\ T\} = \\ \{\mathtt{T}(\text{flu}), \mathtt{T}(\text{headache}), \mathtt{T}(\text{fever}), \mathtt{T}(\text{allergy}), \mathtt{T}(\text{Lime disease})\} = \\ \{\text{Relenza}, \text{Aspirin}, \text{Aspirin}, \text{Claritin}, \text{Doxycycline}\} = B.$$

In particular, $\mathtt{T}$ is an on-to function.

(5). Is $\mathtt{T}$ invertable? If yes, find (the domain, the range, and the graph of) the inverse. If no — or the question itself does not make sense — explain why.

Space for your solution:

The question makes sense because $\mathtt{T}$ is a function. However, this function is not invertible because it is not one-to-one. Indeed,

$$\mathtt{T}(\text{headache}) = \text{Aspirin} = \mathtt{T}(\text{fever}),$$

while

$$\text{headache} \neq \text{fever}.$$

Problem 2. Consider the function with the range $\mathbb{R}$, defined by the formula

$$f(x) = \frac{x^3 - x^2 + x - 1}{x^2 + 2x + 1}$$

for all $x \in \mathbb{R}$, for which the above formula makes sense.

(1). With the usual conventions in effect, what is the domain of the function f?

Space for your solution:

The above formula makes sense if and only if the denominator of the fraction is not zero. The denominator is zero if and only if $x^2 + 2x + 1 = 0 \quad \Leftrightarrow \quad (x+1)^2 = 0 \quad \Leftrightarrow \quad x = -1$. Therefore the domain of the function f is

$$\texttt{Dom}\ f = \{x \in \mathbb{R} : \ x \neq -1\}\,.$$

(2). Find all x-intercepts of the function $f(x)$.

Space for your solution:

The x-intercepts of the function $f(x)$ are the roots of its numerator $x^3 - x^2 + x - 1$. The Rational Roots Theorem states that any rational root of this polynomial will be in the set:

$$\left\{ \frac{\text{divisor of } \ -1}{\text{divisor of } 1} \right\} = \{1, -1\}\,.$$

Substituting these into $x^3 - x^2 + x - 1$ yields the root $x = 1$. Once that root is found, the Polynomial Remainder Theorem of Bézout guarantees divisibility of $x^3 - x^2 + x - 1$ by $x - 1$:

$$\begin{array}{r|l} & x^2 + 0x + 1 \\ \hline x - 1 & x^3 - x^2 + \ x - 1 \\ & x^3 - x^2 \\ \hline & \qquad\quad\ x - 1 \\ & \qquad\quad\ x - 1 \\ \hline & \qquad\qquad\ \ 0 \end{array}$$

Since the resulting quotient $x^2 + 0x + 1 = x^2 + 1$ has no real roots, $x = 1$ is the only real root of the polynomial $x^3 - x^2 + x - 1$. Since $x = 1$ is in the domain of the function $f(x)$, it is the only x-intercept of the function $f(x)$.

(3). Express $f(x)$ as a sum of a polynomial and a proper rational function.

Space for your solution:

The long division:

$$\begin{array}{r|l} & \quad\; x - 3 \\ \hline x^2+2x+1 & x^3 - x^2 + x - 1 \\ & x^3 + 2x^2 + x \\ \hline & \quad -3x^2 + 0x - 1 \\ & \quad -3x^2 - 6x - 3 \\ \hline & \qquad\quad 6x + 2 \end{array}$$

gives:

$$f(x) = \frac{x^3 - x^2 + x - 1}{x^2 + 2x + 1} = x - 3 + \frac{6x + 2}{x^2 + 2x + 1}.$$

(4). Based on the results of the previous sub-problem, determine the oblique asymptote of the function $f(x)$ and the value of x corresponding to the intersection of $f(x)$ with that asymptote.

Space for your solution:

Given that

$$f(x) = \frac{x^3 - x^2 + x - 1}{x^2 + 2x + 1} = x - 3 + \frac{6x + 2}{x^2 + 2x + 1},$$

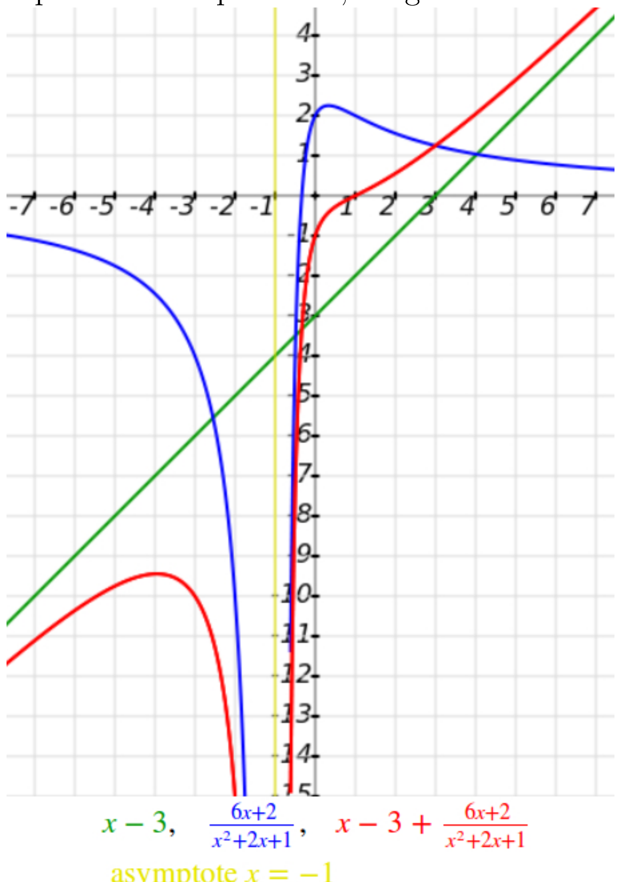
we can conclude that $y = x - 3$ is the oblique asymptote of $f(x)$ and the intersection in question corresponds to the value of x which turns the above fraction into zero:

$$6x + 2 = 0 \quad \Leftrightarrow \quad x = -\tfrac{1}{3}.$$

(5). Sketch the graph of the function $f(x)$.

Space for your solution:

Using the results of the previous sub problems, we get:



Problem 3. In this problem, we will consider functions $3 + \log_2 x$ and $\log_2(3 + x)$.

(1). Solve the equation $3 + \log_2 x = \log_2(3 + x)$ analytically.

Space for your solution:

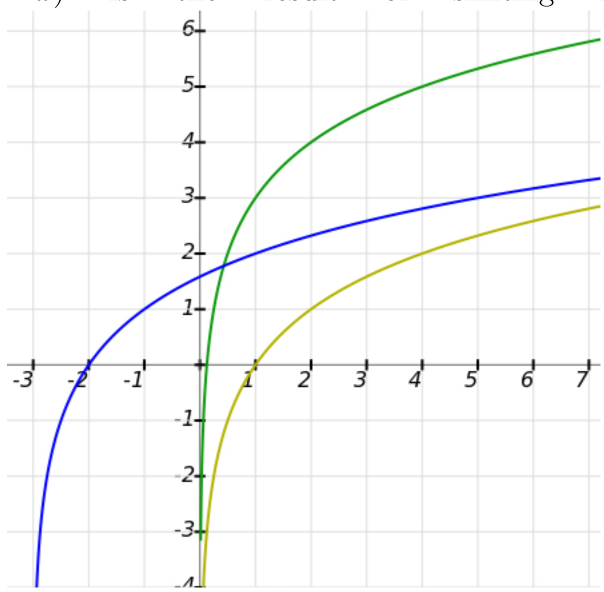
$$3 + \log_2 x = \log_2(3+x) \Leftrightarrow 3 = \log_2(3+x) - \log_2 x \Leftrightarrow \begin{cases} 3 = \log_2 \frac{3+x}{x} \\ 3 + x > 0 \\ x > 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} 2^3 = 2^{\log_2 \frac{3+x}{x}} \\ x > -3 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} 8 = \frac{3+x}{x} \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} 8x = 3 + x \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} 7x = 3 \\ x > 0 \end{cases} \Leftrightarrow x = \frac{3}{7}.$$

(2). Using the technique of graph transformations, sketch the graphs of these functions in the same (x, y)-coordinate system. Is this sketch consistent with your solution of part (1)?

Space for your solution:

The graph of $3 + \log_2 x$ is the result of shifting $\log_2 x$ up by 3, and $\log_2(3 + x)$ is the result of shifting $\log_2 x$ left by 3:



$\log_2(3+x)$, $\log_2 x$, $3 + \log_2 x$.

These two resulting grapsh do intersect at the point consistent with the previous solution $x = \frac{3}{7}$.